Transient Conduction: Spatial Effects

(when lumped capacitance does not apply)

Reminders…

• Homework #5 due Friday

• Friday we start working on the project
  – Scientific reporting
  – Be there!!

• Midterm #1 coming up Wed. October 1
  – Covers chapters 1, 2, 3, 4, 5
  – Make sure you have read the material

• Career fair tomorrow!!
Review: The Biot Number

\[ \text{Bi} = \frac{hL}{k} \]

- If Bi < 0.1 then the lumped capacitance approach can be used
  - Eq. 5.5 to find time to reach a given T
  - Eq. 5.6 to find T after a given time
  - Eq. 5.8a to find total heat gain (loss) for given time

- \( L \) depends on geometry
  - General approach is \( L = \frac{V}{A_s} \)
    - \( L/2 \) for wall with both sides exposed
    - \( r_o/2 \) for long cylinder
    - \( r_o/3 \) for sphere
  - Conservative approach is to use the maximum length
    - \( L \) for wall
    - \( r_o \) for cylinder or sphere (preferred to use this)

Lumped Capacitance Equations

- Time to reach specified temperature (5.5):
  \[ t = \frac{\rho V c}{h A_s} \ln \left( \frac{\theta}{\theta_i} \right) \]

- Temperature after specified time (5.6):
  \[ \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp \left[ -\left( \frac{h A_s}{\rho V c} \right) t \right] \]

- Thermal time constant (5.7):
  \[ \tau_t = \left( \frac{1}{h A_s} \right) (\rho V c) = R_t C_t \]

- Heat transferred during heating (5.8a):
  \[ Q = (\rho V c) \theta_i \left[ 1 - \exp \left( -\frac{t}{\tau_t} \right) \right] \]
About that Pie…
(Can we use a lumped analysis approach?)

Spatial Effects
(When lumped analysis cannot be used)

Dimensionless Variables

Temperature:
\[ \theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} \]

Position:
\[ x^* \equiv \frac{x}{L} \]

Time:
\[ t^* \equiv \frac{\alpha t}{L^2} = \frac{kt}{\rho c_p L^2} \]
Exact Solution for a Plane Wall

- Temperature distribution

\[ \theta^* = \sum_{n=1}^{\infty} C_n \exp \left( -\zeta_n^2 Fo \right) \cos \left( \zeta_n x^* \right) \]

where

\[ C_n = \frac{4 \sin \zeta_n}{2 \zeta_n + \sin (2\zeta_n)} \]

and

\[ t^* \equiv \frac{\alpha t}{L^2} \equiv Fo \]

Note: \( \zeta \) is the eigenvalue (root) of the transcendental equation

Approximate Solution for a Plane Wall

- For plane wall with \( Fo > 0.2 \), temperature distribution

\[ \theta^* = \theta_o^* \cos \left( \zeta_1 x^* \right) \]

where the midpoint \((x = 0)\) temperature \( \theta_o^* \) is

\[ \theta_o^* = C_1 \exp \left( -\zeta_1^2 Fo \right) \]

and \( C_1 \) and \( \zeta_1 \) are found in a table.

- Total heat transfer is:

\[ \frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^* \]
Table 5.1 – $\zeta_1$ and $C_1$ vs. Bi

<table>
<thead>
<tr>
<th>$Bi^*$</th>
<th>$\zeta_1$ (rad)</th>
<th>$C_1$</th>
<th>$\zeta_1$ (rad)</th>
<th>$C_1$</th>
<th>$\zeta_1$ (rad)</th>
<th>$C_1$</th>
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</table>

$^*$Bi = $hL/A$ for the plane wall and $h\sqrt{\varepsilon}/k$ for the infinite cylinder and sphere. See Figure 5.6.

Midplane Temp for Plane Wall

![Midplane temperature as a function of time for a plane wall of thickness 2L [1]. Used with permission.](image)
Temp. Distribution for Plane Wall

Figure 58.2 Temperature distribution in a plane wall of thickness $2L$

Heat Transferred – Plane Wall

Figure 58.3 Internal energy change as a function of time for a plane wall of thickness $2L$. Adapted with permission.
Spatial Effects

• Arises from inadequate solution using lumped capacitance method
  – Temperature gradients are no longer negligible in the medium
• Requires initial and boundary conditions
• Exact solutions involve infinite series
• Approximate solutions use only first term
  – Use Table 5.1 to determine $C_1$ and $\zeta_1$
  – Can use the one-term approximation when $Fo > 0.2$
  – Equations for time, temperature, position, and fraction of total energy transfer for walls, cylinders and spheres

Example – Book Problem 5.39

The 150-mm-thick wall of a gas-fired furnace is constructed of brick ($k = 1.5 \text{ W/m} \cdot \text{K}$, $\rho = 2600 \text{ kg/m}^3$, $c_p = 1000 \text{ J/kg} \cdot \text{K}$) and is well insulated at its outer surface. The wall is at an initial temperature of 20°C when the burners are fired and the inner surface is exposed to products of combustion for which $T_\infty = 950°C$ and $h = 100 \text{ W/m}^2 \cdot \text{K}$.

How long does it take for the outer surface of the wall to reach 750°C?

\[ Bi = \frac{hL}{k} = \frac{(100)(0.15)}{1.5} = 10 \]

• Assume $Fo > 0.2$
• Use the for plane wall:
  \[ \theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \]
• Solve for $t$ via $Fo$:
  \[ Fo = \frac{kt}{\rho c_p L^2} = \frac{\ln(\theta_o^* / C_1)}{\zeta_1^2} \]
Table 5.1 – $\zeta_1$ and $C_1$ vs. Bi

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<thead>
<tr>
<th>Bi</th>
<th>Plane Wall $\zeta_1$ (rad)</th>
<th>$C_1$</th>
<th>Infinite Cylinder $\zeta_1$ (rad)</th>
<th>$C_1$</th>
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9.0  1.4149  1.2598  2.1566  1.5611  2.8044  1.9106
10.0 1.4289  1.2620  2.1795  1.5677  2.8363  1.9249
20.0 1.4961  1.2699  2.2881  1.5919  2.9857  1.9781

Problem 5.39

Temperature Distribution over Time

Temperature, °C

Dimensionless location, x/L

- t=0 s
- t=10,000 s
- t=20,000 s
- t=33,800 s
Approximate Solutions for Cylinders and Spheres

- Similar approach as for plane wall
  - NOTE: For cylinders and spheres, use $r_o$ for calculation of Bi and use that Bi to look up values in the table

- Cylinder:
  \[
  \theta^* = \theta_o^* J_0(\zeta_1 r^*)
  \]
  \[
  \theta_o^* = C_1 \exp(-\zeta_1^2 Fo)
  \]
  with centerline $T$:
  \[
  \frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1)
  \]
  and total energy transfer:
  \[
  \frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3} [\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)]
  \]

...and for spheres

- Sphere:
  \[
  \theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*)
  \]
  with center $T$:
  \[
  \theta_o^* = C_1 \exp(-\zeta_1^2 Fo)
  \]
  and total energy transfer:
  \[
  \frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3} [\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)]
  \]