QUESTION

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1. (20 pts) The Lumped Capacitance Method
   - **(a)** List and describe the implications of the two major assumptions of the lumped capacitance method. *(6 pts)*
   - ***(b)*** Define the Biot number by equations and words. *(4 pts)*
   - ***(c)*** Describe what happens when Bi << 1. *(3 pts)*
   - ***(d)*** Describe what happens when Bi >> 1. *(3 pts)*
   - **(e)** Define the Fourier number by equation. *(4 pts)*

ANSWER

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a. The basis of the lumped capacitance method is that the temperature of the solid in question is spatially uniform at any given instant during a transient process. The key point of this assumption implies that temperature gradients within the solid are negligible. The second major assumption of the lumped capacitance method is that that resistance to conduction within the solid is small compared to the resistance to heat transfer between the solid and its surroundings. It is important to know that this case implies infinite thermal conductivity, which is practically impossible, but necessary nonetheless as an extension of Fourier's law for visualizing and solving problems like these.

b. The Biot number, Bi, is a dimensionless parameter that relates thermal conductivity and surface convective heat transfer. From a practical view, it is the ratio of a materials ability to transfer heat convectively to its ability to transfer heat conductively. It is defined in the following manner.

$$\frac{R_{cond}}{R_{conv}} = \frac{hL}{k} \equiv Bi$$

c. When Bi << 1, the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer. This means that the assumption of a uniform temperature distribution is reasonable. See page 260, Incropera, et.el.

d. When Bi >> 1, the temperature difference across the solid is much larger than that between the surface of the fluid, temperature gradients become more significant:

$$T = T(x, t)$$

.

e. Fourier number:

$$Fo \equiv \frac{\alpha t}{L_c^2}$$

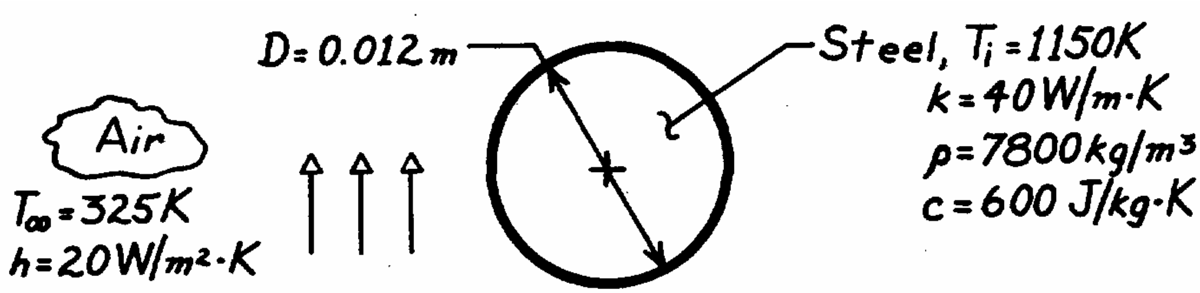
α = thermal diffusivity, t = time, L_c = characteristic length

PROBLEM 5.5

KNOWN: Diameter and initial temperature of steel balls cooling in air.

FIND: Time required to cool to a prescribed temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible radiation effects, (2) Constant properties.

ANALYSIS: Applying Eq. 5.10 to a sphere ($L_c = r_o/3$),

$$Bi = \frac{hL_c}{k} = \frac{h(r_o/3)}{k} = \frac{20\ W/m^2 \cdot K\ (0.002m)}{40\ W/m \cdot K} = 0.001.$$

Hence, the temperature of the steel remains approximately uniform during the cooling process, and the lumped capacitance method may be used. From Eqs. 5.4 and 5.5,

$$t = \frac{\rho V c_p}{hA_s} \ln\frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho\left(\pi D^3/6\right)c_p}{h\pi\ D^2} \ln\frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{7800 kg/m^3 (0.012m) 600 J/kg \cdot K}{6 \times 20\ W/m^2 \cdot K} \ln\frac{1150 - 325}{400 - 325}$$

$$t = 1122\ s\ = 0.312h$$ <

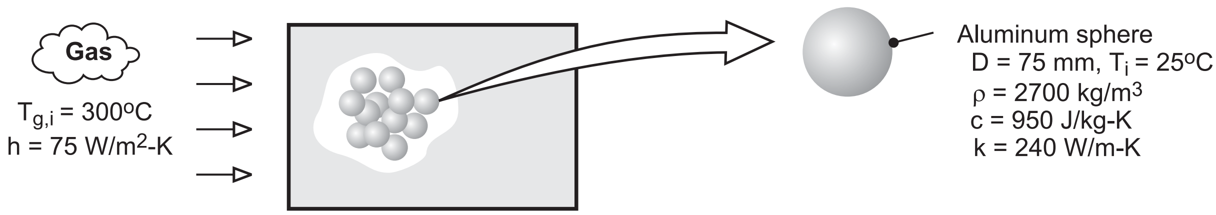
COMMENTS: Due to the large value of T_i, radiation effects are likely to be significant during the early portion of the transient. The effect is to shorten the cooling time.

PROBLEM 5.12

KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere to acquire 90% of maximum possible thermal energy and the corresponding center temperature. Potential advantage of using copper in lieu of aluminum.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute Bi = $h(r_o/3)/k = 75\ W/m^2\cdot K\ (0.025m)/150\ W/m\cdot K = 0.013 < 0.1$. Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time. From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$\frac{Q}{\rho c V \theta_i} = 0.90 = 1 - \exp\left(-t/\tau_t\right)$$

where $\tau_t = \rho Vc/hA_s = \rho Dc/6h = 2700\,kg/m^3 \times 0.075m \times 950\,J/kg\cdot K/6 \times 75\,W/m^2\cdot K = 427s.$ Hence,

$$t = -\tau_t \ln(0.1) = 427s \times 2.30 = 984s$$ <

From Eq. (5.6), the corresponding temperature at any location in the sphere is

$$T(984s) = T_{g,i} + \left(T_i - T_{g,i}\right)\exp\left(-6ht/\rho Dc\right)$$

$$T(984s) = 300°C - 275°C\ \exp\left(-6 \times 75\,W/m^2\cdot K \times 984s/2700\,kg/m^3 \times 0.075m \times 950\,J/kg\cdot K\right)$$

$$T(984)s = 272.5°C$$ <

Obtaining the density and specific heat of copper from Table A-1, we see that $(\rho c)_{Cu} \approx 8900\ kg/m^3 \times 400\ J/kg\cdot K = 3.56 \times 10^6\ J/m^3\cdot K > (\rho c)_{Al} = 2.57 \times 10^6\ J/m^3\cdot K$. Hence, for an equivalent sphere diameter, the copper can store approximately 38% more thermal energy than the aluminum.

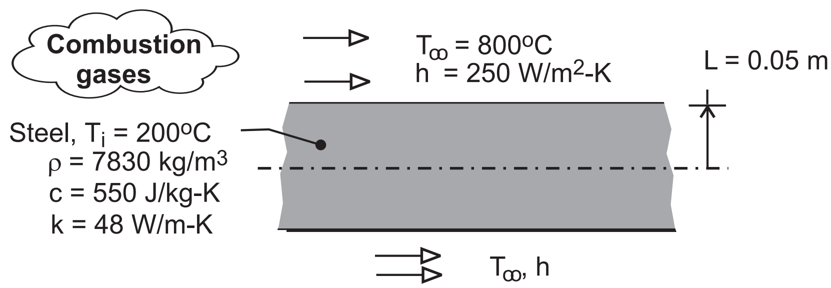
COMMENTS: Before the packed bed becomes fully charged, the temperature of the gas decreases as it passes through the bed. Hence, the time required for a sphere to reach a prescribed state of thermal energy storage increases with increasing distance from the bed inlet.

PROBLEM 5.37

KNOWN: Thickness, properties and initial temperature of steel slab. Convection conditions.

FIND: Heating time required to achieve a minimum temperature of 550°C in the slab.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Negligible radiation effects, (3) Constant properties.

ANALYSIS: With a Biot number of $hL/k = (250\ W/m^2\cdot K \times 0.05m)/48\ W/m\cdot K = 0.260$, a lumped capacitance analysis should not be performed. At any time during heating, the lowest temperature in the slab is at the midplane, and from the one-term approximation to the transient thermal response of a plane wall, Eq. (5.41), we obtain

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = \frac{(550-800)°C}{(200-800)°C} = 0.417 = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

With $\zeta_1 \approx 0.488$ rad and $C_1 \approx 1.0396$ from Table 5.1 and $\alpha = k/\rho c = 1.115\times10^{-5} m^2/s$,

$$-\zeta_1^2\left(\alpha t/L^2\right) = \ln(0.401) = -0.914$$

$$t = \frac{0.914\,L^2}{\zeta_1^2\alpha} = \frac{0.841(0.05m)^2}{(0.488)^2\,1.115\times10^{-5} m^2/s} = 861s$$ <

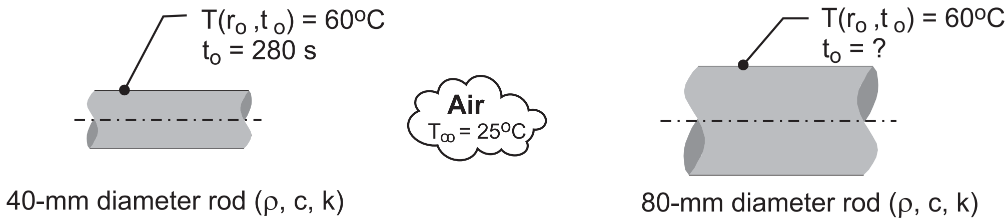
COMMENTS: The surface temperature at t = 861s may be obtained from Eq. (5.40b), where $\theta^* = \theta_o^* \cos\left(\zeta_1 x^*\right) = 0.417 \cos(0.488\ rad) = 0.368$. Hence, $T(L, 792s) \equiv T_s = T_\infty + 0.368(T_i - T_\infty)$ $= 800°C - 221°C = 579°C$. Assuming a surface emissivity of ε = 1 and surroundings that are at $T_{sur} = T_\infty = 800°C$, the radiation heat transfer coefficient corresponding to this surface temperature is $h_r = \varepsilon\sigma(T_s + T_{sur})\left(T_s^2 + T_{sur}^2\right) = 205\ W/m^2\cdot K$. Since this value is comparable to the convection coefficient, radiation is not negligible and the desired heating will occur well before t = 861s.

PROBLEM 5.54

KNOWN: Long rods of 40 mm- and 80-mm diameter at a uniform temperature of 400°C in a curing oven, are removed and cooled by forced convection with air at 25°C. The 40-mm diameter rod takes 280 s to reach a *safe-to-handle* temperature of 60°C.

FIND: Time it takes for a 80-mm diameter rod to cool to the same safe-to-handle temperature. Comment on the result? Did you anticipate this outcome?

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional radial (cylindrical) conduction in the rods, (2) Constant properties, and (3) Convection coefficient same value for both rods.

PROPERTIES: Rod (*given*): ρ = 2500 kg/m^3, c = 900 J/kg·K, k = 15 W/m·K.

ANALYSIS: Not knowing the convection coefficient, the Biot number cannot be calculated to determine whether the rods behave as spacewise isothermal objects. Using the relations from Section 5.6, Radial Systems with Convection, for the infinite cylinder, Eq. 5.50, evaluate $Fo = \alpha t / r_o^2$, and knowing $T(r_o, t_o)$, a trial-and-error solution is required to find $Bi = h\, r_o/k$ and hence, h. Using the *IHT Transient Conduction* model for the *Cylinder*, the following results are readily calculated for the 40-mm rod. With t_o = 280 s,

$$Fo = 4.667 \qquad Bi = 0.264 \qquad h = 197.7\ W/m^2 \cdot K$$

For the 80-mm rod, with the foregoing value for h, with $T(r_o, t_o)$ = 60°C, find

$$Bi = 0.528 \qquad Fo = 2.413 \qquad t_o = 579\ s$$ <

COMMENTS: (1) The time-to-cool, t_o, for the 80-mm rod is slightly more than twice that for the 40-mm rod. Did you anticipate this result? Did you believe the times would be proportional to the diameter squared?

(2) The simplest approach to explaining the relationship between t_o and the diameter follows from the lumped capacitance analysis, Eq. 5.13, where for the same θ/θ_i, we expect $Bi \cdot Fo_o$ to be a constant. That is,

$$\frac{h \cdot r_o}{k} \times \frac{\alpha\, t_o}{r_o^2} = C$$

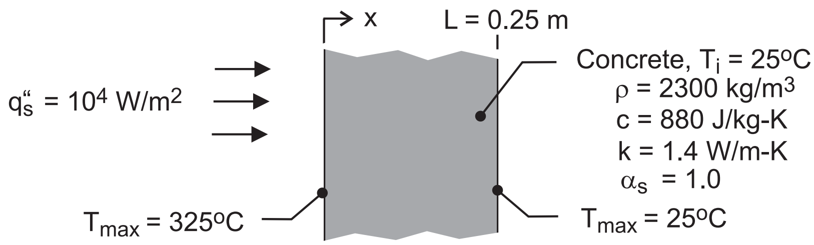
yielding $t_o \sim r_o$ (not r_o^2).

PROBLEM 5.79

KNOWN: Thickness, initial temperature and thermophysical properties of concrete firewall. Incident radiant flux and duration of radiant heating. Maximum allowable surface temperatures at the end of heating.

FIND: If maximum allowable temperatures are exceeded.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in wall, (2) Validity of semi-infinite medium approximation, (3) Negligible convection and radiative exchange with the surroundings at the irradiated surface, (4) Negligible heat transfer from the back surface, (5) Constant properties.

ANALYSIS: The thermal response of the wall is described by Eq. (5.59)

$$T(x,t) = T_i + \frac{2\,q_o''(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_o'' x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where, $\alpha = k/\rho c_p = 6.92\times10^{-7}\,m^2/s$ and for $t = 30\ min = 1800s$, $2q_o''(\alpha t/\pi)^{1/2}/k = 284.5\,K$. Hence, at x = 0,

$$T(0, 30\ min) = 25°C + 284.5°C = 309.5°C < 325°C$$ <

At $x = 0.25m$, $\left(-x^2/4\alpha t\right) = -12.54$, $q_o'' x/k = 1,786K$, and $x/2(\alpha t)^{1/2} = 3.54$. Hence,

$$T(0.25m,\ 30\,min) = 25°C + 284.5°C\left(3.58\times10^{-6}\right) - 1786°C\times(\sim 0) \approx 25°C$$ <

Both requirements are met.

COMMENTS: The foregoing analysis is conservative since heat transfer at the irradiated surface due to convection and net radiation exchange with the environment have been neglected. If the emissivity of the surface and the temperature of the surroundings are assumed to be ε = 1 and T_{sur} = 298K, radiation exchange at T_s = 309.5°C would be $q_{rad}'' = \varepsilon\sigma\left(T_s^4 - T_{sur}^4\right) = 6,080\ W/m^2\cdot K$, which is significant (~ 60% of the prescribed radiation).

SAPPHIRE ROD PROBLEM

As always, we first check the Biot number:

$$Bi = \frac{h\frac{r_o}{2}}{k} = \frac{(1600\ W/m^2 \cdot K)\left(\frac{0.02\ m}{2}\right)}{(22.3\ W/m \cdot K)} = \underline{0.72}$$

↑ Interpolated from Table A.2 for the midpoint temp = 550 K

→ Since $Bi > 0.1$ we cannot use the lumped capacitance approach.

For Table 5.1 we need to use r_o instead of $r_o/2$ for the Biot number calculation.

$$Bi = \frac{h r_o}{k} = 1.43$$

We also need to confirm that the one-term approximation is an okay simplification.

$$Fo = \frac{\alpha t}{r_o^2} = \frac{(5.26 \times 10^{-6}\ m^2/s)(35\ s)}{(0.02\ m)^2} = \underline{0.46}$$

→ α was interpolated from Table 5.1 using $T = 550\ K$

→ Since $Fo > 0.2$ we can use the one-term approximation.

Continued...

SAPPHIRE ROD, CONT.

→ Interpolating C_1 and ζ_1 from Table 5.1 for $Bi = 1.43$ we get:

$$C_1 = 1.2636$$
$$\zeta_1 = 1.4036$$

So the idea is to see how much energy is removed during the 35 seconds of cooling and to then identify the average temperature decrease associated with this energy loss. Equation 5.51 (6th Edition) for energy transfer uses the term Θ_0^* so first we should calculate that.

$$\Theta_0^* = C_1 \exp\left(-\zeta_1^2 F_0\right) = 1.2636 \exp\left[-(1.4036)^2\, 0.46\right]$$
$$= \underline{0.511}$$

$$Q/Q_0 = 1 - \frac{2\Theta_0^*}{\zeta_1} J_1(\zeta_1)$$

NOTE! J_1 is the Bessel function of the first kind, Appendix B.4

$$= 1 - \frac{2(0.511)}{1.4036}(0.5425)$$

$$= 0.605$$

→ We can calculate $Q = \rho V c_p \Delta T$ but it is not necessary. Since heat capacity is constant, Q/Q_0 is proportional to $\frac{T_i - T}{T_i - T_\infty}$ for the whole rod once equilibrated.

$$T = T_i - 0.605(T_i - T_\infty) = 800 - 0.605(500) = \boxed{498\,K}$$

SANDSTONE LEDGE PROBLEM

Consider this a semi-infinite solid with a constant surface temperature. (case 1 on Page 286)

Sandstone properties:

$$\rho = 2150 \text{ kg/m}^3$$
$$k = 2.9 \text{ W/m·K}$$
$$c_p = 745 \text{ J/kg·K}$$

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

← $k/\rho c_p$

$$= \frac{(2.9 \text{ W/m·K})(15 - 40)}{\left[\pi (2.9 \text{ W/m·K}) / (2150 \frac{\text{kg}}{\text{m}^3})(745 \frac{\text{J}}{\text{kg·K}})\right]^{1/2}} \left(\frac{1}{t^{1/2}}\right)$$

→ Integrate this gives

$$\int_0^{300} 30{,}399\, t^{-1/2}\, dt$$

$$= \boxed{1{,}053{,}000 \text{ J/m}^2}$$

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