LES of Turbulent Flows: Lecture 4
(ME EN 7960-003)

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Scale Separation

- We discussed LES in a very generic way to this point:
  - Resolve only the largest energy containing scales
  - Model the small "universal" scales

- Formally, how is this accomplished?
  - Using a **low-pass filter** (i.e., removes small scale motions)

- Our goal for the low pass filter:
  - Attenuate (smooth) **high frequency** (high wavenumber/small scale) turbulence smaller than a characteristic scale Δ while leaving **low frequency** (low wavenumber/large scale) motions unchanged.

Filtering

- **Filtering** (Saguat chapter 2; Pope chapter 13.2):
  -The formal (mathematical) LES filter is a convolution filter defined for a quantity $\phi(\vec{x},t)$ in physical space as

$$\widetilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$$

  - $G\equiv$ the convolution kernel of the chosen filter
  - G is associated with a characteristic cutoff scale Δ (also called the filter width)

  - Taking the Fourier transform of $\widetilde{\phi}(\vec{x})$ (dropping the *t* for simplicity)

$$F\{\widetilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{x}} \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta})G(\vec{\zeta})d\vec{\zeta}d\vec{x}$$

Here we will use Pope's notation for the Fourier transform: $F\{\phi(x)\} = \int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx$

Convolution

- we can define a new variable: $\vec{r} = \vec{x} - \vec{\zeta}$ and change the order of integration

$$F\{\widetilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\vec{k}(\vec{r}+\vec{\zeta})} \phi(\vec{r}) G(\vec{\zeta}) d\vec{\zeta} d\vec{r}$$

Note that $d\vec{r} = d\vec{x}$ because $\vec{\zeta} \neq f(\vec{x})$ and addition of exponents is multiplication =>

$$F\{\widetilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{r}} \phi(\vec{r}) d\vec{r} \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{\zeta}} G(\vec{\zeta}) d\vec{\zeta}$$

$$\boxed{= F\{\phi(\vec{x})\} F\{G(\vec{\zeta})\}}$$

Sagaut writes this as:

$$\widehat{\widetilde{\phi}}(\vec{k}, \omega) = \widehat{\phi}(\vec{k}, \omega) \widehat{G}(\vec{k}, \omega)$$

where the hat ($\frown$) denotes a Fourier coefficient.

- $\widehat{G}$ is the transfer function associated with the filter kernel G
Recall that a transfer function is the wavespace (Fourier) relationship between the **input** and **output** of a linear system.

Decomposition into resolved and subfilter components

- Just as G is associated with a filter scale Δ (filter width), $\widehat{G}$ is associated with a cutoff wavenumber k_c.

- In a similar manner to Reynold's decomposition, we can use the filter function to decompose the velocity field into resolved and unresolved (or subfilter) components

$$\phi(\vec{x},t) = \widetilde{\phi}(\vec{x},t) + \phi'(\vec{x},t)$$

total — resolved — subfilter

- **Fundamental properties** of "proper" LES filters:

  -The filter **shouldn't change** the value of **a constant** (a):
  $$\int_{-\infty}^{\infty} G(\vec{x})d\vec{x} = 1 \;\Rightarrow\; \widetilde{a} = a$$

  - **Linearity**: $\widetilde{\phi + \zeta} = \widetilde{\phi} + \widetilde{\zeta}$

  (this is satisfied automatically for a convolution filter)

  - **Commutation with differentiation**:

$$\widetilde{\frac{\partial \phi}{\partial \vec{x}}} = \frac{\partial \widetilde{\phi}}{\partial \vec{x}}$$

LES and Reynold's Operators

- In the general case, LES filters that verify these properties are not Reynolds operators

 -Recall for a Reynolds operator (average) defined by $\langle\ \rangle$

- $\langle a\phi\rangle = a\langle\phi\rangle$
- $\langle\phi'\rangle = 0$
- $\langle\phi + \zeta\rangle = \langle\phi\rangle + \langle\zeta\rangle$
- $\langle\langle\phi\rangle\rangle = \langle\phi\rangle$
- $\langle\frac{\partial\phi}{\partial\vec{x}}\rangle = \frac{\partial\langle\phi\rangle}{\partial\vec{x}}$

- For our LES filter, in general (using Sagauts shorthand $\int_{-\infty}^{\infty}\phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta} = G\star\phi$):
  - $\tilde{\tilde{\phi}} = G\star G\star\phi = G^2\star\phi \neq \tilde{\phi} = G\star\phi$
  - $\widetilde{\phi'} = G\star(\phi - G\star\phi) \neq 0$

- For an LES filter a twice filtered variable is not equal to a single filtered variable as it is for a Reynolds average.

- Likewise, the filtered subfilter scale component is not equal to zero

Differential Filters

- **Differential filters** are a subclass of convolution filter

  - The filter kernel is the Green's function associated to an inverse linear differential operator

  - Recall, the Green's function of a linear differential operator *L* satisfies $L(x)G_r(x,s) = \delta(x-s)$ and can be used to find the solution of inhomogeneous differential equations subject to certain boundary conditions.

- The inverse linear differential operator *J* is defined by:

$$\phi = J(\widetilde{\phi}) = J(G \star \phi)$$

which can be expanded to: $= \widetilde{\phi} + \theta \frac{\partial \widetilde{\phi}}{\partial t} + \Delta_\ell \frac{\partial \widetilde{\phi}}{\partial x_\ell} + \Delta_{\ell m} \frac{\partial^2 \widetilde{\phi}}{\partial x_\ell \partial x_m}$

Effectively we need to invert the above equation to define the filter kernel G. See Sagaut pgs 20-21 and the references contained therein for more information.

- Differential filters are not used much in practice and can be considered an "advanced" topic in LES

Typical LES filters

Common (or classic) LES filters:

• Box or top-hat filter: (equivalent to a local average)

$$G(x-\zeta) = \begin{cases} \frac{1}{\Delta} \text{ if } |x-\zeta| \le \frac{\Delta}{2} \\ 0 \quad \text{otherwise} \end{cases}$$

transfer function

$$\hat{G}(k) = \frac{sin(k\Delta/2)}{k\Delta/2}$$

• Gaussian filter: (γ typically = 6)

$$G(x-\zeta) = \frac{\gamma}{\pi\Delta^2}^{\frac{1}{2}} exp\left(\frac{-\gamma|x-\zeta|^2}{\Delta^2}\right)$$

transfer function

$$\hat{G}(k) = exp\left(\frac{-\Delta^2 k^2}{4\gamma}\right)$$

• Spectral or sharp cutoff filter:

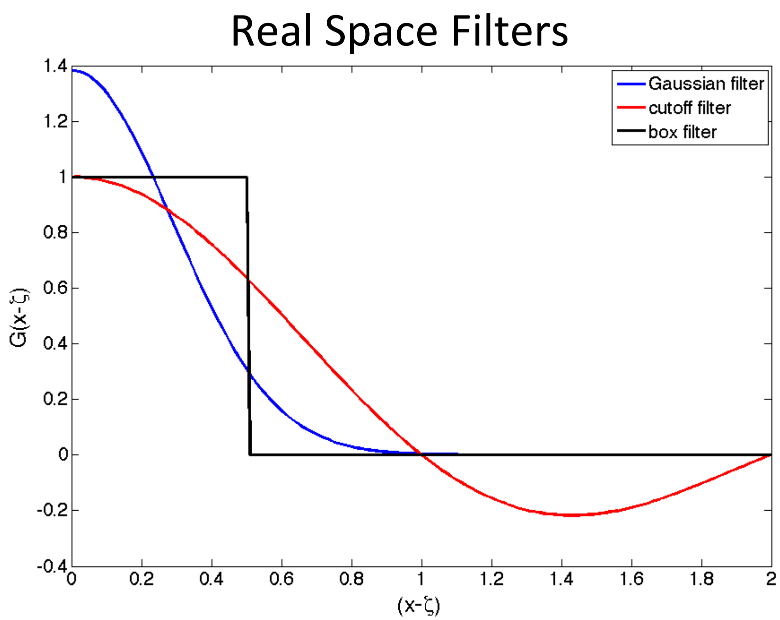
$$G(x-\zeta) = \frac{sin(k_c(x-\zeta))}{k_c(x-\zeta)}$$

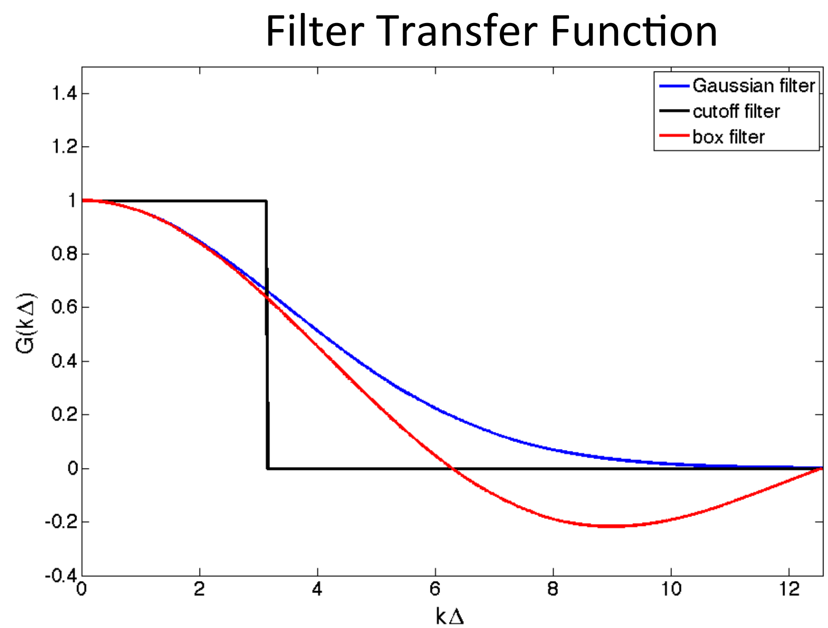
transfer function

$$\hat{G}(k) = \begin{cases} 1 \text{ if } |k| \le k_c \\ 0 \quad \text{otherwise} \end{cases}$$

(recall that k_c is our characteristic wavenumber cutoff)

LES Filters and their transfer functions





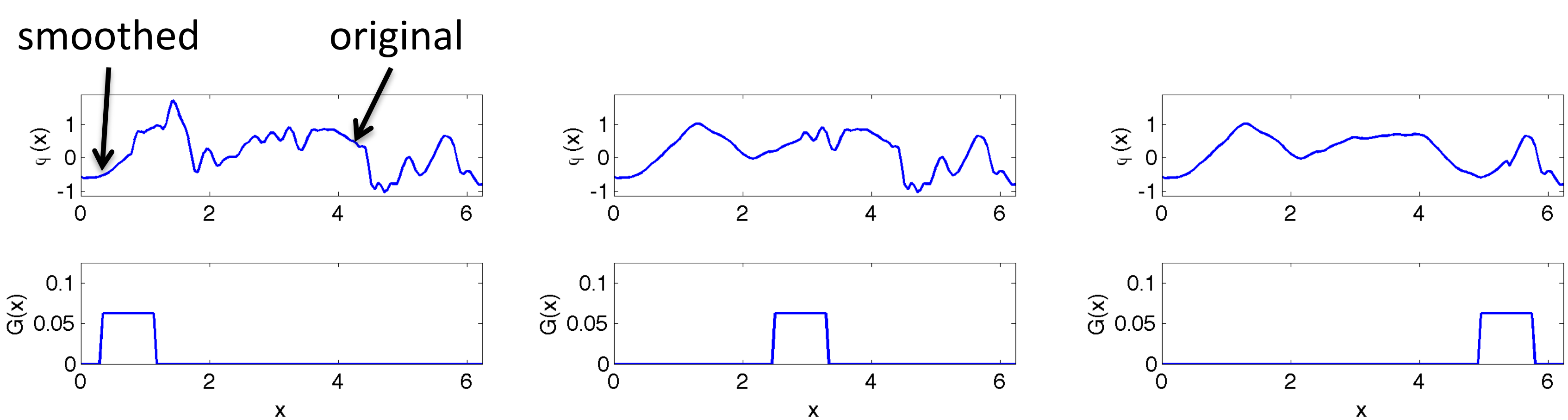
Only the Gaussian filter is local in both real and wave space

Convolution Example

- We defined convolution of two functions as: $\widetilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$
- How can we interpret this relation?

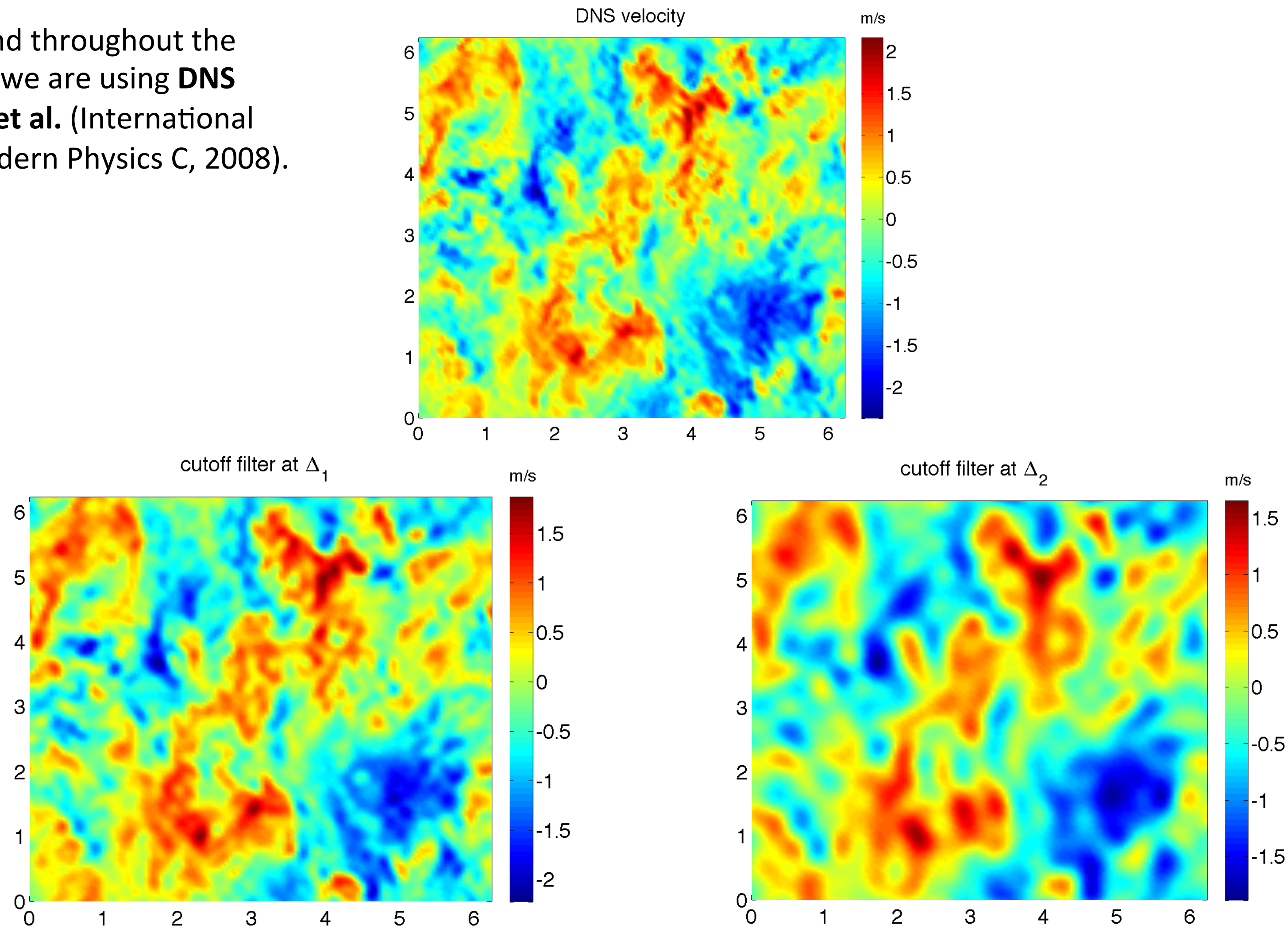
  -G, our filter kernel 'moves' along our function ϕ smoothing it out (provided it is a low-pass filter):

  - Example using a box filter applied in real space (see mfile conv_example.m):



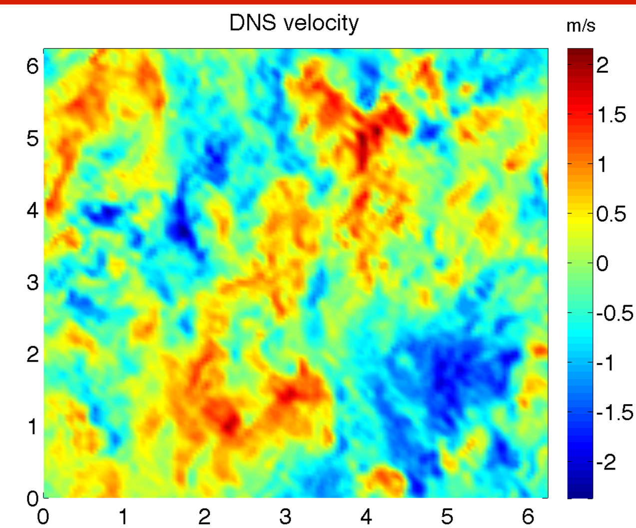
Filtering Turbulence (real space, cutoff filter)

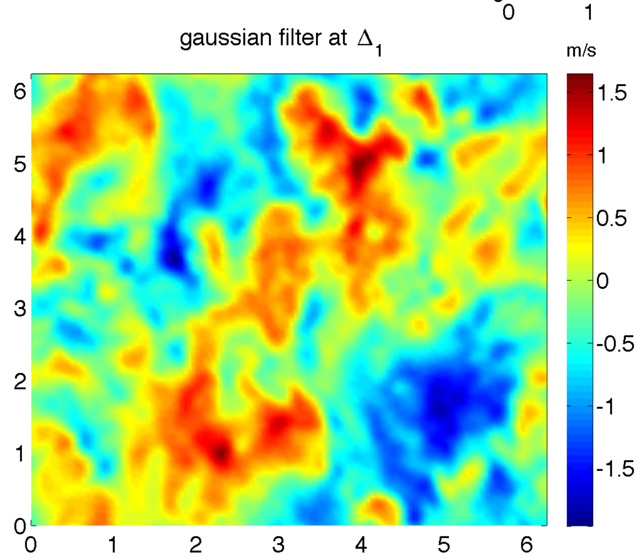
Note: here (and throughout the presentation) we are using **DNS data from Lu et al.** (International Journal of Modern Physics C, 2008).

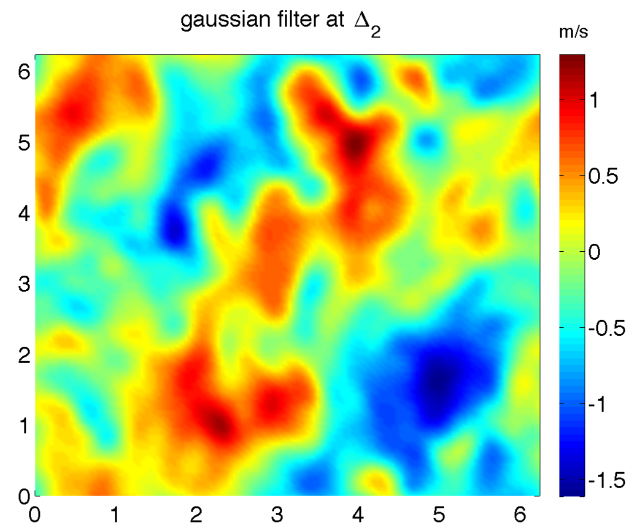


Filtering Turbulence (real space, Gaussian filter)

Note: here (and throughout the presentation) we are using **DNS data from Lu et al.** (International Journal of Modern Physics C, 2008).







Filtering Turbulence (real space, box filter)

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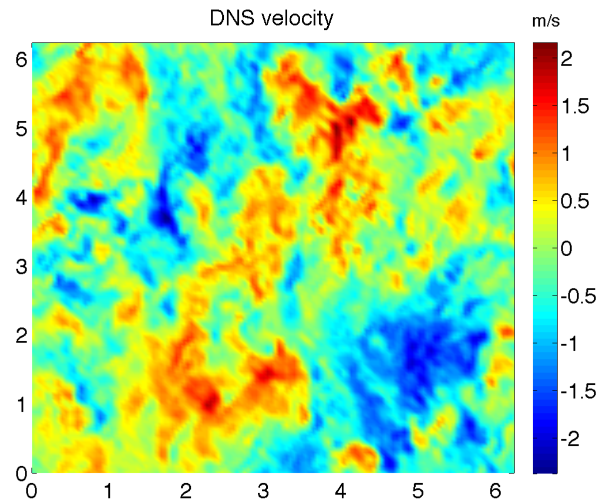

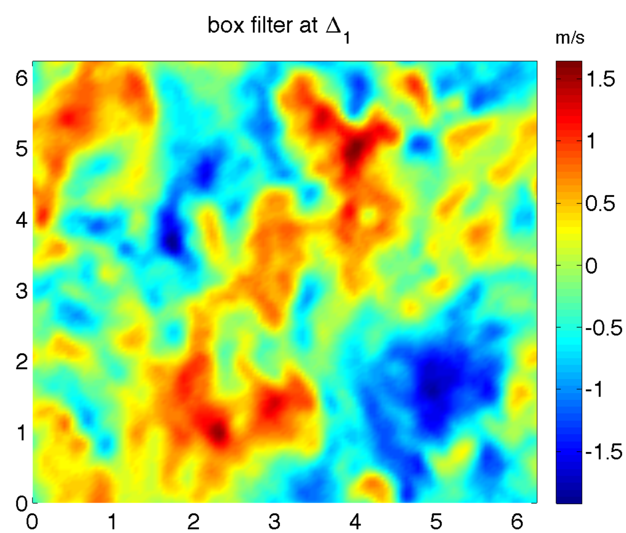

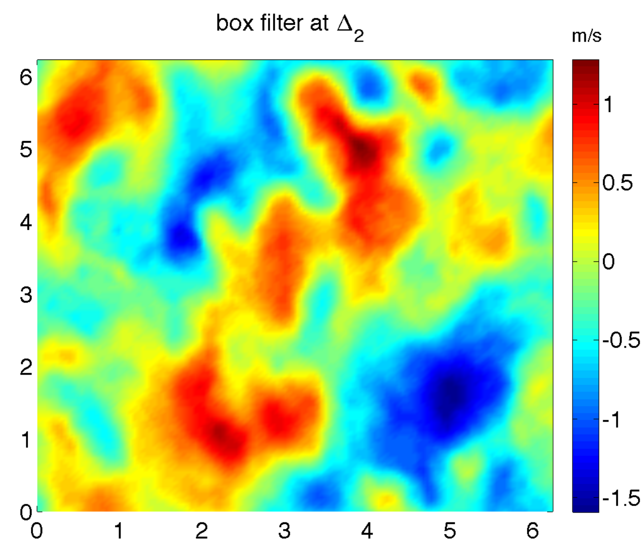

Filtering Turbulence (real space)

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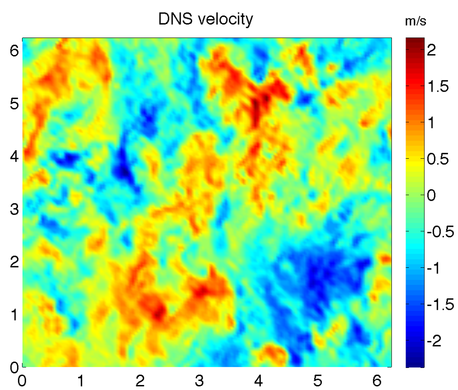
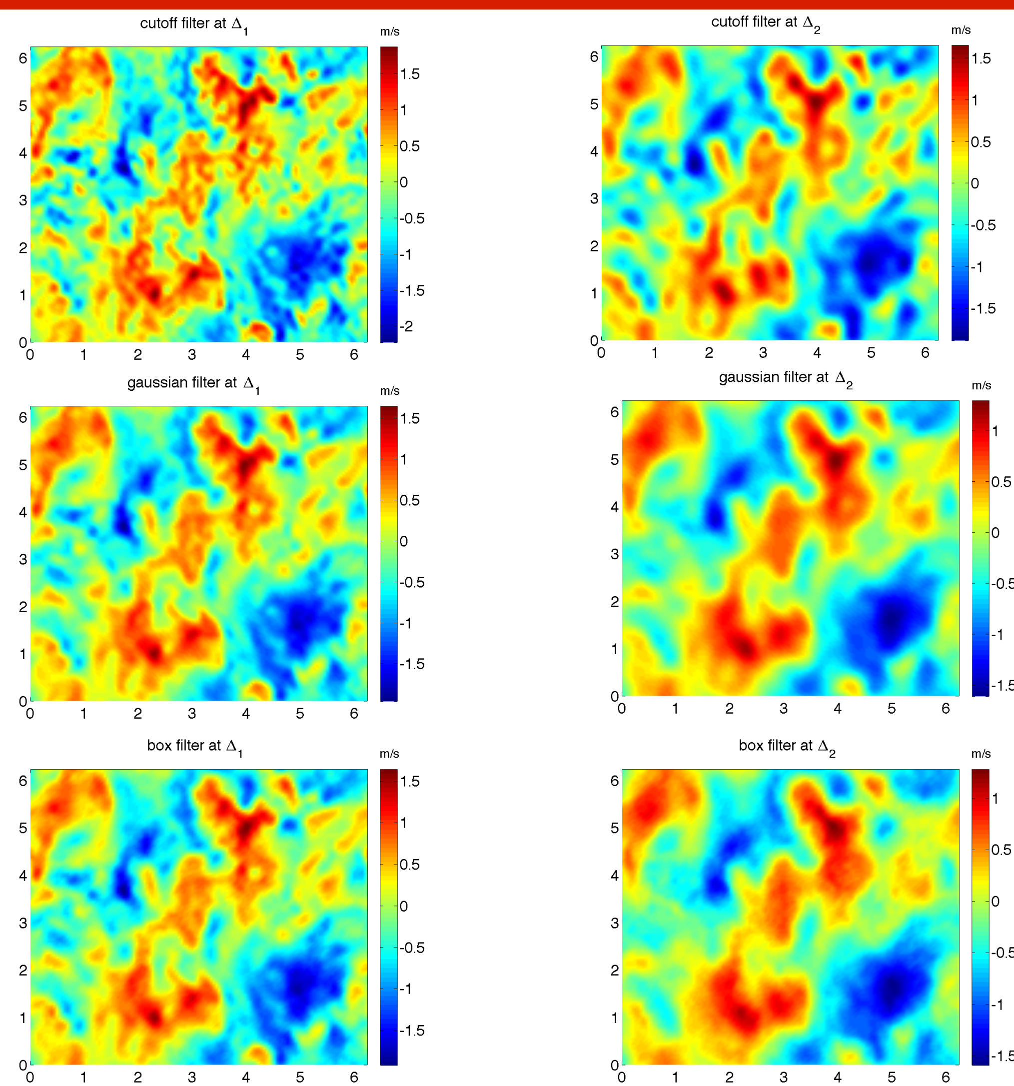
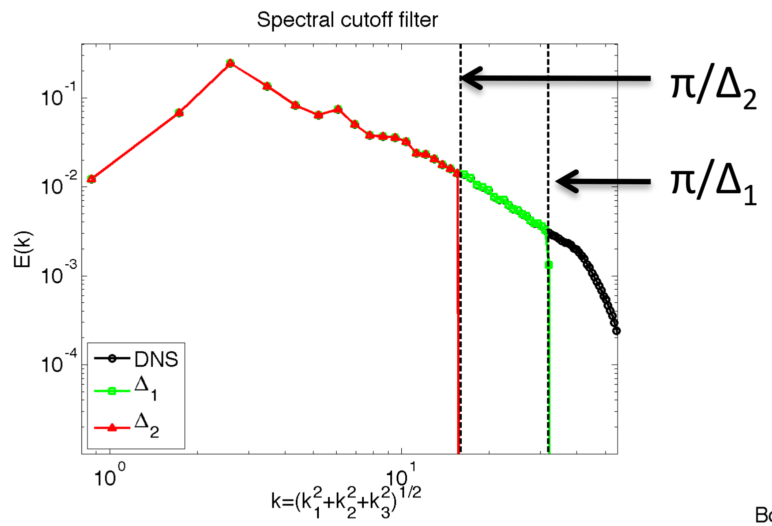
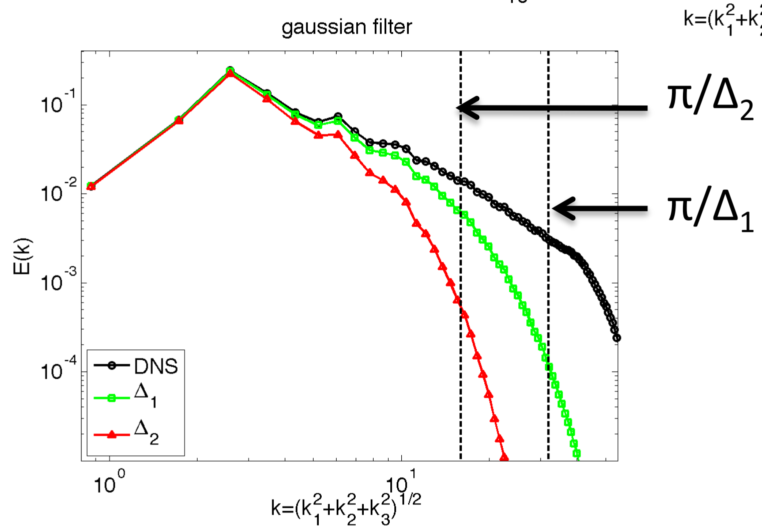
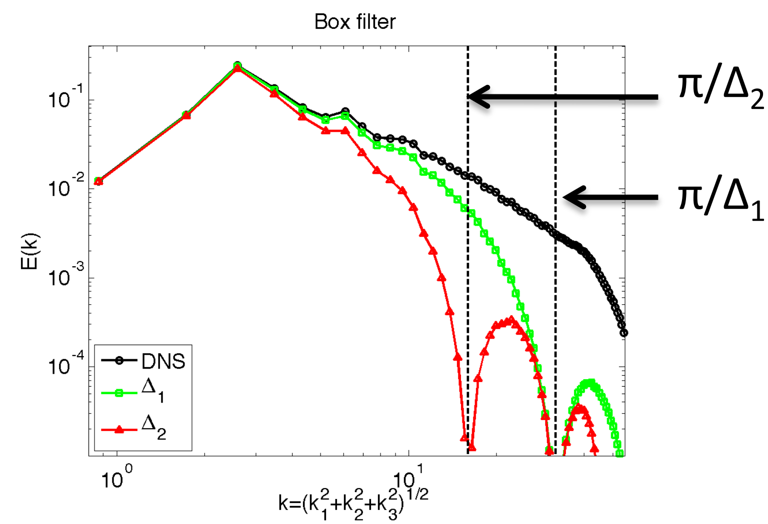




Filtering Turbulence (wave space)

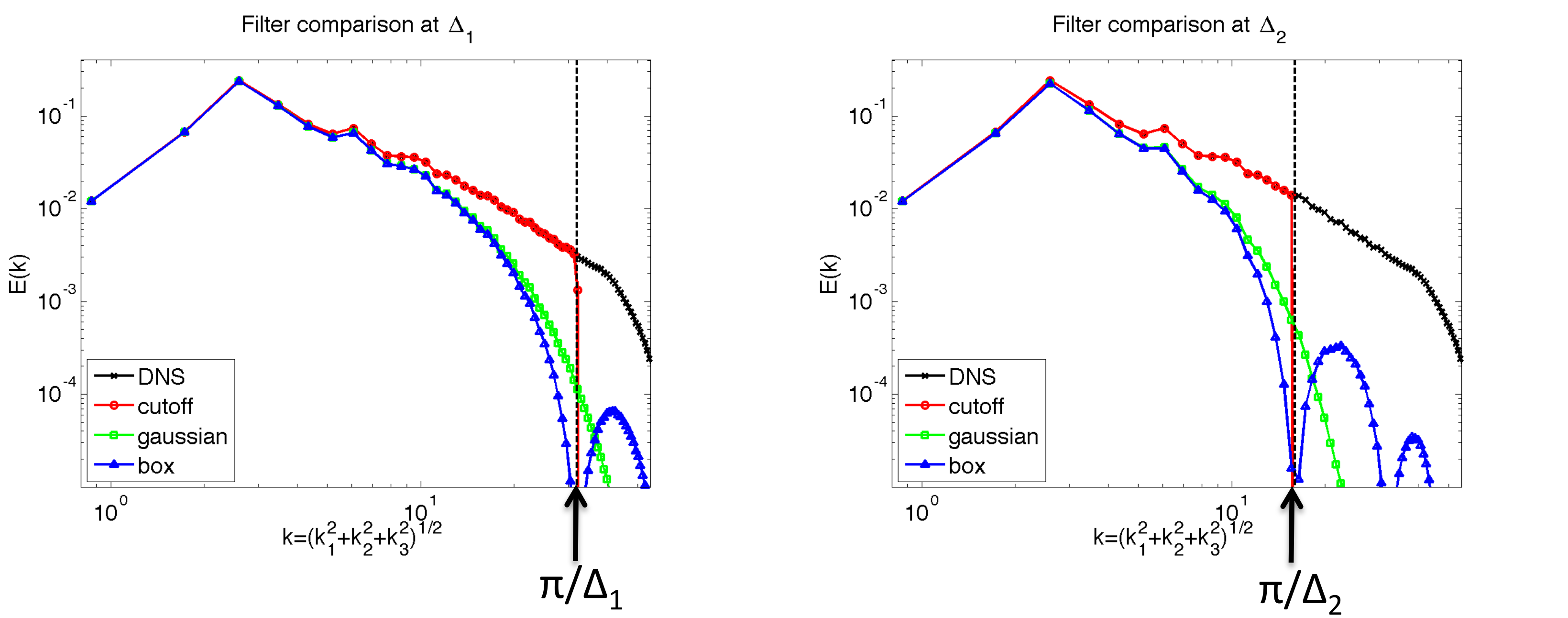






Filtering Turbulence (wave space)

Comparison between different filters

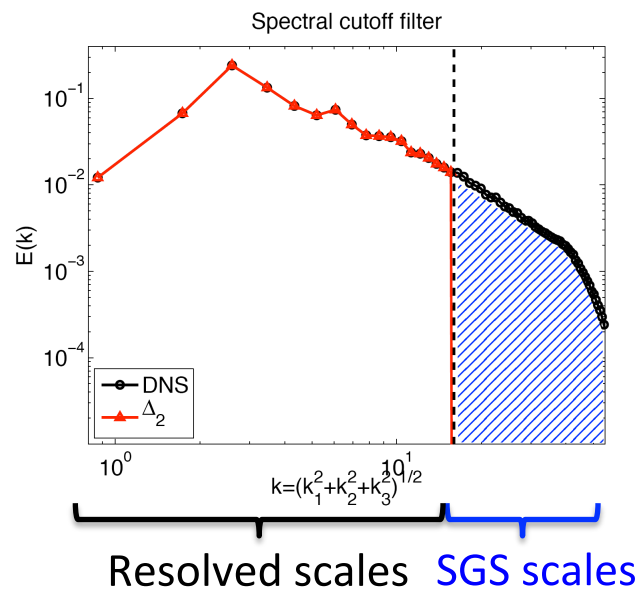


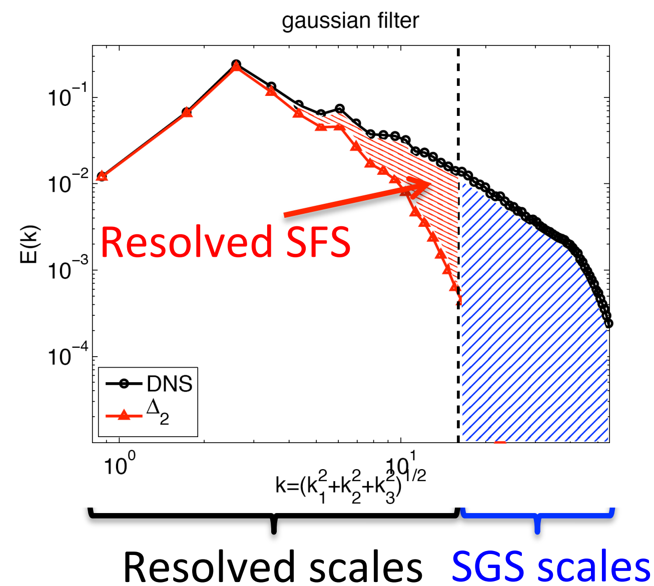
Decomposition of Turbulence for real filters

The LES filter can be used to decompose the velocity field into resolved and subfilter scale (SFS) components

$$\phi(\vec{x},t) = \widetilde{\phi}(\vec{x},t) + \phi'(\vec{x},t)$$

We can use our filtered DNS fields to look at how the choice of our filter kernel affects this separation in wavespace





The Gaussian filter (or box filter) does not have as compact of support in wavespace as the cutoff filter. This results in attenuation of energy at scales larger than the filter scale. The scales affected by this attenuation are referred to as **Resolved SFSs**.