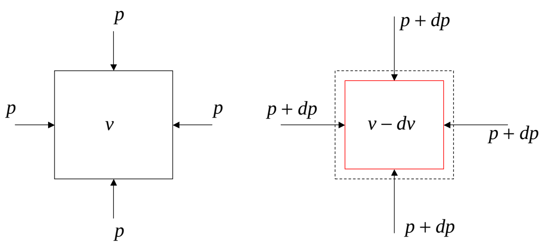
Introduction to Compressible Flow

$$\frac{D\rho}{Dt} \neq 0$$

The density of a gas changes significantly along a streamline

Compressible Flow

Definition of Compressibility: the fractional change in volume of the fluid element per unit change in pressure



Compressible Flow

1. Mach Number: $M = \dfrac{V}{c} = \dfrac{\text{local velocity}}{\text{speed of sound}}$
2. Compressibility becomes important for High Speed Flows where $M > 0.3$
   - $M < 0.3$ – Subsonic & incompressible
   - $0.3 < M < 0.8$ – Subsonic & compressible
   - $0.8 < M < 1.2$ – transonic flow – shock waves appear mixed subsonic and sonic flow regime
   - $1.2 < M < 3.0$ - Supersonic – shock waves are present but NO subsonic flow
   - M > 3.0 – Hypersonic Flow, shock waves and other flow changes are very strong

Compressible Flow

3. Significant changes in velocity and pressure result in density variations throughout a flow field

4. Large Temperature variations result in density variations.

As a result we now have two new variables we must solve for:
T & ρ
We need 2 new equations.
We will solve: mass, linear momentum, energy and an equation of state.

Important Effects of Compressibility on Flow

1. *Choked Flow* – a flow rate in a duct is limited by the sonic condition
2. *Sound Wave/Pressure Waves* – rise and fall of pressure during the passage of an acoustic/sound wave. The magnitude of the pressure change is very small.
3. *Shock Waves* – nearly discontinuous property changes in supersonic flow. (Explosions, high speed flight, gun firing, nuclear explosion)
4. A pressure ratio of 2:1 will cause sonic flow

Applications

1. *Nozzles and Diffusers and converging diverging nozzles*
2. *Turbines, fans & pumps*
3. *Throttles – flow regulators, an obstruction in a duct that controls pressure drop.*
4. *One Dimensional Isentropic Flow – compressible pipe flow.*

Approach

- Control volume approach
- Steady, One-dimension, Uniform Flow
- Additional Thermodynamics Concepts are needed
- Restrict our analysis to ideal gases

Thermodynamics

- Equation of State – Ideal Gas Law

$$p = \rho RT$$

$$R = \frac{R_u}{M_m} = \frac{\text{Universal Gas Constant}}{\text{Molecular mass of air}} = \frac{8314\ \text{J/(kmol}\cdot\text{K)}}{28.97\text{kg/kmol}} = 287\ \text{J/(kg}\cdot\text{K)}$$

Temperature is absolute and the specific volume is (volume per unit mass):

$$v = \frac{1}{\rho}$$

Thermodynamics – Internal Energy & Enthalpy

- Internal Energy – individual particle kinetic energy. Summation of molecular vibrational and rotational energy.

$$\tilde{u} = \tilde{u}(v, T)$$

$$d\tilde{u} = \left(\frac{\partial \tilde{u}}{\partial T}\right)_v dT + \left(\frac{\partial \tilde{u}}{\partial v}\right)_T dv$$

- For an ideal gas $\tilde{u} = \tilde{u}(T)$

$$d\tilde{u} = c_v dT$$

- Recall from our integral form of the Energy Equation for Enthalpy of an ideal gas: $h = \tilde{u} + pv$

$$h = h(T)$$

$$dh = c_p dT$$

Thermodynamics – Internal Energy & Enthalpy

$$h = \tilde{u} + pv$$
$$h = \tilde{u} + RT \longrightarrow \frac{p}{\rho} = RT$$
$$dh = d\tilde{u} + RdT$$

Substituting:

$$dh = c_p dT \qquad d\tilde{u} = c_v dT$$

$$dh = d\tilde{u} + RdT$$
$$c_p dT = c_v dT + RdT$$
$$c_p = c_v + R$$
$$c_p - c_v = R = const$$

Thermodynamics – Internal Energy & Enthalpy

Define the ratio of specific heats: $k \equiv \dfrac{c_p}{c_v} = const$

Then,

$$c_p = \frac{kR}{k-1}$$
$$c_v = \frac{R}{k-1}$$

For Air:
c_p = 1004 J/kg-K
k = 1.4

The 2nd Law of Thermodynamics & Isentropic Processes

We define entropy by:

$$ds = \left(\frac{\delta Q}{T}\right)_{rev}$$

Combining the 1st and 2nd Laws gives us Gibb's Equation

$$Tds = dh - \frac{dp}{\rho}$$

$$Tds = c_p dT - \frac{dp}{\rho} \longleftarrow dh = c_p dT$$

$$\int_1^2 ds = c_p \int_1^2 \frac{dT}{T} - R\int_1^2 \frac{dp}{p} \longleftarrow \frac{1}{\rho T} = \frac{R}{p}$$

The 2nd Law of Thermodynamics & Isentropic Processes

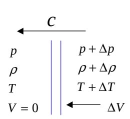
$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

For an Isentropic process: adiabatic and reversible
We get the following power law relationship

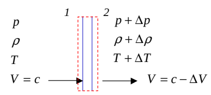
$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

Control Volume Analysis of a Finite Strength Pressure Wave

Moving Wave of Frontal Area A



Stationary Wave
Reference frame moving with wave



Steady State Continuity Equation (Solve for the *induced velocity* ΔV):

$$0 = \int_{CS} \rho\left(\vec{V}\bullet\hat{n}\right)dA = -\int_1 \rho c dA + \int_2 (\rho+\Delta\rho)(c-\Delta V)dA$$

$$\rho c A = (\rho+\Delta\rho)(c-\Delta V)A$$

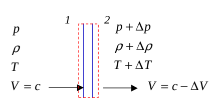
$$\rho c = c(\rho+\Delta\rho) - \Delta V(\rho+\Delta\rho)$$

$$\Delta V = c\frac{\Delta\rho}{\rho+\Delta\rho} \quad \text{(A)}$$

The Speed of sound (c) is the rate of propagation of a pressure wave of infinitesimal strength through a still fluid.

Control Volume Analysis of a Finite Strength Pressure Wave

Steady State Momentum Equation:
(Find Δp and c)



$$\sum F_x = \int_{CS} \rho V_x\left(\vec{V}\bullet\hat{n}\right)dA = \dot{m}(V_2 - V_1)$$

$$pA - (p+\Delta p)A = \rho c A(c - \Delta V - c)$$

$$\Delta p = \rho c \Delta V \quad \text{(B)}$$

Now combine A & B and solve for the speed of sound:

$$c^2 = \frac{\Delta p}{\Delta\rho}\frac{\rho+\Delta\rho}{\rho} = \frac{\Delta p}{\Delta\rho}\left(1+\frac{\Delta\rho}{\rho}\right)$$

$$c^2 = \frac{\partial p}{\partial\rho} \quad \text{in the limit of } \Delta\rho \to 0$$

Small Amplitude moderate frequency waves are isentropic and $\longrightarrow$ $\frac{p}{\rho^k} = const$

Control Volume Analysis of a Finite Strength Pressure Wave

Calculating the Speed of Sound for an ideal gas:

$$\frac{p}{\rho^k} = const$$

$$\frac{\partial p}{\partial \rho} = k\frac{p}{\rho} \longrightarrow c = \sqrt{k\frac{p}{\rho}} = \sqrt{kRT}$$

$$\boxed{c = \sqrt{kRT}}$$

For Air:

$$k = \frac{c_p}{c_v} \approx 1.4$$

$$R = 287\ \mathrm{J/(kg \cdot K)}$$

Typical Speeds of Sound

Fluid	c (m/s)
Gases:	
H_2	1,294
Air	340
Liquids:	
Water	1,490
Ethyl Alcohol	1,200

Data From White 2003

Example 1: Speed of sound calculation

Determine the speed of sound in Argon (Ar) at 120 °C. MW = 40 kg/kmol:

$$c = \sqrt{kRT}$$

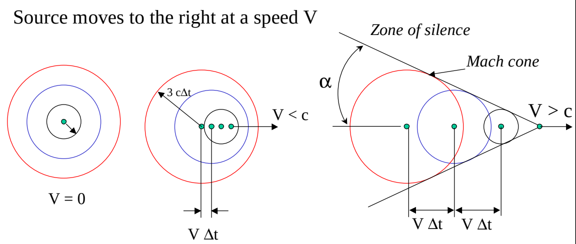
$$R = \frac{R_u}{M_m} = \frac{8314\ \mathrm{J/(kmol \cdot K)}}{40\mathrm{kg/kmol}} = 207.9\mathrm{J/(kg \cdot K)}$$

$$k = \frac{c_p}{c_v} \approx 1.668$$

$$c = \sqrt{1.668(207.9\mathrm{J/kgK})(393K)} = 318.8\mathrm{ms}^{-1}$$

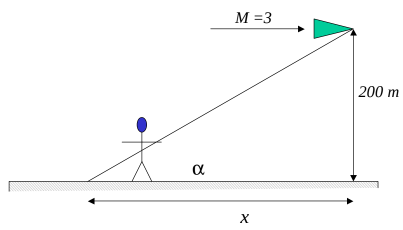
Movement of a sound source and wave propagation

Source moves to the right at a speed V



$$\sin\alpha = \frac{V}{c} = \frac{1}{M}$$

Example 2: a needle nose projectile traveling at a speed of M=3 passes 200m above an observer. Find the projectiles velocity and determine how far beyond the observer the projectile will first be heard



Example 2: a needle nose projectile traveling at a speed of M=3 passes 200m above an observer. Find the projectiles velocity and determine how far beyond the observer the projectile will first be heard

$$c = \sqrt{kRT} = \sqrt{1.4(287)(300)} = 347.2\text{m/s}$$

$$V = Mc = 3(347.2) = 1041.6\text{m/s}$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{3}\right) = 19.5^o$$

$$\tan\alpha = \frac{200m}{x}$$

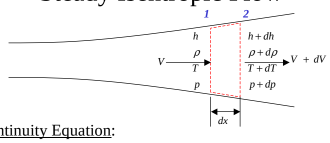
$$\boxed{x = \frac{200m}{\tan 19.5} = 565m}$$

Steady Isentropic Flow – Control Volume Analysis

Applications where the assumptions of steady, uniform, isentropic flow are reasonable:

1. Exhaust gasses passing through the blades of a turbine.
2. Diffuser near the front of a jet engine
3. Nozzles on a rocket engine
4. A broken natural gas line

Steady Isentropic Flow



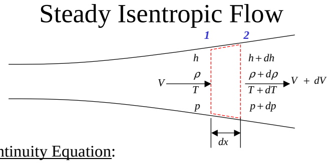
Steady State <u>Continuity Equation</u>:

$$0 = \int_{CS} \rho\left(\vec{V} \bullet \hat{n}\right) dA = -\rho_1 V_1 A_1 + \rho_2 V_2 A_2$$

$$\rho VA = (\rho + d\rho)(V + dV)(A + dA)$$

$$\rho VA = \rho AV + \rho VdA + VAd\rho + Vd\rho dA + \rho AdV + \rho dAdV + Ad\rho dV + d\rho dAdV$$

Steady Isentropic Flow



Steady State <u>Continuity Equation</u>:

$$0 = \int_{CS} \rho\left(\vec{V} \bullet \hat{n}\right) dA = -\rho_1 V_1 A_1 + \rho_2 V_2 A_2$$

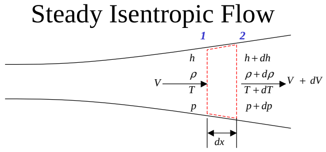
$$\rho VA = (\rho + d\rho)(V + dV)(A + dA)$$

$$\rho VA = \rho AV + \rho VdA + VAd\rho + \cancelto{\sim 0}{Vd\rho dA} + \rho AdV + \cancelto{\sim 0}{\rho dAdV} + \cancelto{\sim 0}{Ad\rho dV} + \cancelto{\sim 0}{d\rho dAdV}$$

$$\boxed{0 = \frac{dA}{A} + \frac{d\rho}{\rho} + \frac{dV}{V}}$$

Only retain 1st order differential terms & divide By ρVA

Steady Isentropic Flow



Steady State <u>Energy Equation</u> with
1 inlet & 1 exit:

$$\frac{\dot{Q} - \dot{W}_s}{\dot{m}} = \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) + (\tilde{u} + pv)_2 - (\tilde{u} + pv)_1$$

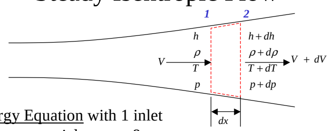
Neglecting potential energy and recalling: $h = \tilde{u} + pv$

$$\frac{\dot{Q} - \dot{W}_s}{\dot{m}} = \frac{V_2^2 - V_1^2}{2} + h_2 - h_1$$

Assuming and ideal gas:

$$\frac{\dot{Q} - \dot{W}_s}{\dot{m}} = \frac{V_2^2 - V_1^2}{2} + c_p(T_2 - T_1)$$

Steady Isentropic Flow



Steady State <u>Energy Equation</u> with 1 inlet & 1 exit, neglecting potential energy & assuming Isentropic duct flow:

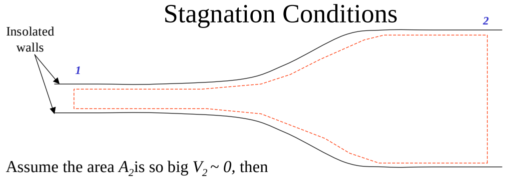
$$\frac{V_2^2}{2} + h_2 = \frac{V_1^2}{2} + h_1$$

Assuming and ideal gas:

$$\frac{V_2^2}{2} + c_p T_2 = \frac{V_1^2}{2} + c_p T_1$$

$$\frac{V_2^2}{2} + \frac{k}{k-1} R T_2 = \frac{V_1^2}{2} + \frac{k}{k-1} R T_1$$

Stagnation Conditions



Assume the area A_2 is so big $V_2 \sim 0$, then

$$h_2 = \frac{V_1^2}{2} + h_1 = h_o \longrightarrow \text{Stagnation enthalpy}$$

Similarly, as we adiabatically bring a fluid parcel to zero velocity there is a corresponding increase in temperature

$$\frac{V_2^2}{2} + c_p T_2 = \frac{V_1^2}{2} + c_p T_1$$

$$\boxed{T_o = \frac{V^2}{2c_p} + T} \longrightarrow \text{Stagnation Temperature}$$

Stagnation Conditions – maximum velocity

$$T_o = \frac{V^2}{2c_p} + T \qquad (+)$$

If the temperature, T is taken taken down to absolute zero, then (+) can be solved for the maximum velocity:

$$V_{\max} = \sqrt{2 c_p T_o}$$

No higher velocity is possible unless energy is added to the flow through heat transfer or shaft work.

Stagnation Conditions – Mach number relations

Recall, that the Mach number is defined as: $M = \frac{V}{c}$

$$T_o = \frac{V^2}{2c_p} + T$$

For Ideal gases:

$$\frac{T_o}{T} = \frac{V^2}{T\,2c_p} + 1 \longrightarrow c_p T = \underbrace{\left(\frac{kR}{k-1}\right)}_{c_p} T = \underbrace{kRT}_{c^2}\,\frac{1}{k-1}$$

$$\frac{T_o}{T} = \frac{k-1}{2}\frac{V^2}{c^2} + 1 = \frac{k-1}{2}M^2 + 1$$

$$\boxed{\frac{T_o}{T} = \frac{k-1}{2}M^2 + 1}$$

Stagnation Conditions – Isentropic pressure & density relationships

$$\frac{T_o}{T} = \frac{k-1}{2}M^2 + 1$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{k}{k-1}} = \left(\frac{k-1}{2}M^2 + 1\right)^{\frac{k}{k-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{k-1}} = \left(\frac{k-1}{2}M^2 + 1\right)^{\frac{1}{k-1}}$$

Critical Values: conditions when $M = 1$

$$\frac{T^*}{T_o} = \left(\frac{2}{k+1}\right)$$

$$\frac{p^*}{p_o} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}$$

$$\frac{c^*}{c_o} = \left(\frac{2}{k+1}\right)^{\frac{1}{2}}$$

Critical Values: conditions when $M = 1$

For Air $k = 1.4$

$$\frac{T^*}{T_o} = \left(\frac{2}{k+1}\right) = 0.8333$$

$$\frac{p^*}{p_o} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} = 0.5283$$

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} = 0.9129$$

$$\frac{c^*}{c_o} = \left(\frac{2}{k+1}\right)^{\frac{1}{2}} = 0.9129$$

In all isentropic flow, all critical values are constant.

Critical Values: conditions when $M = 1$

Critical Velocity: is the speed of sound c^*

$$\frac{c^*}{c_o} = \left(\frac{2}{k+1}\right)^{\frac{1}{2}}$$

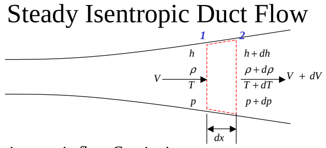
$$V^* = c^* = \sqrt{kRT^*} = c_o\left(\frac{2}{k+1}\right)^{\frac{1}{2}} = \left(\frac{2kRT_o}{k+1}\right)^{\frac{1}{2}}$$

Example 3: Stagnation Conditions

Air flows adiabatically through a duct. At point *1* the velocity is 240 m/s, with T_1 = 320K and p_1 = 170kPa. Compute

(a) T_o
(b) P_o
(c) r_o
(d) M
(e) V_{max}
(f) V^*

Steady Isentropic Duct Flow



Recall, for Steady isentropic flow Continuity:

$$(\dagger)\quad 0 = \frac{dA}{A} + \frac{d\rho}{\rho} + \frac{dV}{V} \longrightarrow \frac{dA}{A} = -\frac{d\rho}{\rho} - \frac{dV}{V}$$

For compressible, isentropic flow the momentum equation is:

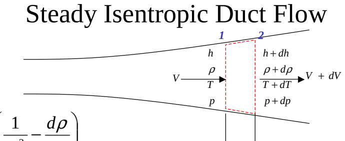
$$(*)\quad 0 = \frac{dp}{\rho} + \frac{dV^2}{2} = \frac{dp}{\rho} + VdV \longrightarrow$$

Bernoulli's Equation!
neglecting gravity

Substitute (†) into (*)

$$\frac{dA}{A} = -\frac{d\rho}{\rho} + \frac{dp}{\rho V^2} = \frac{dp}{\rho}\left(\frac{1}{V^2} - \frac{d\rho}{dp}\right)$$

Steady Isentropic Duct Flow



$$\frac{dA}{A} = \frac{dp}{\rho}\left(\frac{1}{V^2} - \frac{d\rho}{dp}\right)$$

Recall that the speed of sound is: $c^2 = \frac{\partial p}{\partial \rho}$

$$\frac{dA}{A} = \frac{dp}{\rho}\left(\frac{1}{V^2} - \frac{1}{c^2}\right) = \frac{dp}{\rho V^2}\left(1 - \frac{V^2}{c^2}\right)$$

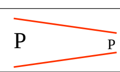
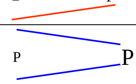
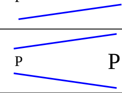
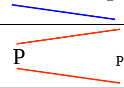
Substituting the Mach number: $M = \frac{V}{c}$

$$\boxed{\frac{dA}{A} = \frac{dp}{\rho V^2}\left(1 - M^2\right)} \longrightarrow$$

Describes how the pressure behaves in nozzles and diffusers under various flow conditions

Nozzle Flow Characteristics

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left(1 - M^2\right)$$

1.	*Subsonic Flow*: $M < 1$ and $dA < 0$, then $dP < 0$: indicating a decrease in pressure in a converging channel.	
2.	*Supersonic Flow:* $M > 1$ and $dA < 0$, then $dP > 0$: indicating an increase in pressure in a converging channel.	
3.	*Subsonic Flow*: $M < 1$ and $dA > 0$, then $dP > 0$: indicating an increase in pressure in a diverging channel.	
4.	*Supersonic Flow:* $M > 1$ $dA > 0$, then $dP < 0$: indicating a decrease in pressure in a diverging channel.	

Steady Isentropic Duct Flow – Nozzles Diffusers and Converging Diverging Nozzles

(††) $\dfrac{dA}{A} = \dfrac{dp}{\rho V^2}\left(1 - M^2\right) \longrightarrow$ Describes how the pressure behaves in nozzles and diffusers under various flow conditions

Recall, the momentum equation here is:

$$0 = \frac{dp}{\rho} + VdV \longrightarrow \frac{dp}{\rho} = -VdV \quad (**)$$

Now substitute (**) into (††) :

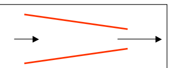
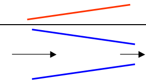
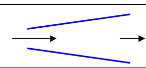
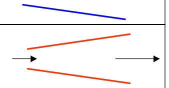
$$\boxed{\frac{dA}{A} = \frac{dV}{V}\left(M^2 - 1\right)}$$

Or,

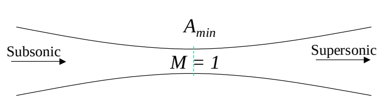
$$\frac{dA}{dV} = \frac{A}{V}\left(M^2 - 1\right)$$

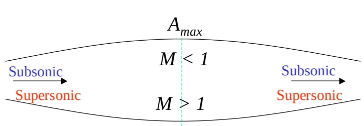
Nozzle Flow Characteristics

$$\frac{dA}{A} = \frac{dV}{V}\left(M^2 - 1\right)$$

1. *Subsonic Flow*: $M < 1$ and $dA < 0$, then $dV > 0$: indicating an accelerating flow in a converging channel.	
2. *Supersonic Flow:* $M > 1$ and $dA < 0$, then $dV < 0$: indicating an decelerating flow in a converging channel.	
3. *Subsonic Flow*: $M < 1$ and $dA > 0$, then $dV < 0$: indicating an decelerating flow in a diverging channel.	
4. *Supersonic Flow:* $M > 1$ $dA > 0$, then $dV > 0$: indicating an accelerating flow in a diverging channel.	

Converging-Diverging Nozzles

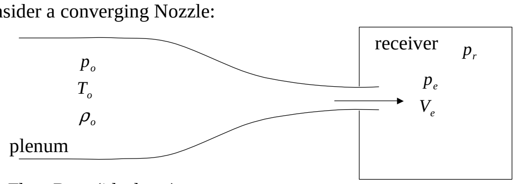




Flow can not be sonic

Choked Flow – The maximum possible mass flow through a duct occurs when it's throat is at the sonic condition

Consider a converging Nozzle:



Mass Flow Rate (ideal gas):

$$\dot{m} = \rho VA = \frac{p}{RT}VA \qquad M = \frac{V}{c} = \frac{V}{\sqrt{kRT}}$$

$$\dot{m} = \frac{p}{RT}M\sqrt{kRT}\,A = p\sqrt{\frac{k}{RT}}MA$$

$$\boxed{\dot{m} = p\sqrt{\frac{k}{RT}}MA}$$

Choked Flow

Mass Flow Rate (ideal gas):

$$\dot{m} = p\sqrt{\frac{k}{RT}}MA$$

Recall, the stagnation pressure and Temperature ratio and substitute:

$$\frac{p_o}{p} = \left(\frac{k-1}{2}M^2+1\right)^{\frac{k}{k-1}} \qquad \frac{T_o}{T} = \frac{k-1}{2}M^2+1$$

$$\dot{m} = p_o\sqrt{\frac{k}{RT_o}}MA\left(1+\frac{k-1}{2}M^2\right)^{\frac{k+1}{2(1-k)}}$$

If the critical area (A*) is where M=1:

$$\dot{m} = p_o A^*\sqrt{\frac{k}{RT_o}}\left(\frac{k+1}{2}\right)^{\frac{k+1}{2(1-k)}}$$

The critical area Ratio is:

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{2+(k-1)M^2}{k+1}\right)^{\frac{k+1}{2(k-1)}}$$