

6 VORTICITY DYNAMICS

As mentioned in the introduction, turbulence is rotational and characterized by large fluctuations in vorticity. In this section we would like to identify some of the mechanisms of vorticity transport and highlight their effects on the dynamics of turbulent flows. This will lead into a more general discussion of the energy cascade and some famous hypotheses concerning the behavior of turbulence.

6.1 Vorticity Equation

The equation for the transport of vorticity can be obtained by taking the curl of the momentum equation. The result of this operation is:

$$\frac{D\omega}{Dt} = -\omega(\nabla \cdot \mathbf{V}) + \frac{(\nabla\rho \times \nabla p)}{\rho^2} + (\omega \cdot \nabla)\mathbf{V} + \frac{1}{Re}(\nabla^2\omega) \quad (6.1)$$

or, in Cartesian tensor notation

$$\frac{\partial\omega_i}{\partial t} + \frac{\partial u_j\omega_i}{\partial x_j} = -\omega_i \frac{\partial u_j}{\partial x_j} + \epsilon_{ijk} \frac{\partial\rho}{\partial x_j} \frac{\partial p}{\partial x_k} / \rho^2 + \omega_j \frac{\partial}{\partial x_j} u_i + \frac{1}{Re} \frac{\partial^2\omega_i}{\partial x_j \partial x_j} \quad (6.2)$$

Eq. 6.1 or 6.2 describes the transport and generation of vorticity for a general three-dimensional flow. Four different physical mechanisms can be identified above that affect the vorticity transport. Below each of them is briefly discussed.

Expansion The first term on the rhs of Eq. 6.1 represents the effects of expansion on the vorticity field. In a flow that is expanding, $\nabla \cdot \mathbf{V}$ is a positive quantity. This term would then result in a decrease in the magnitude of vorticity due to the minus sign in front of this term. This term can play a major role in the vorticity dynamics of combustion or reacting flows, as the combustion heat release can result in large changes in the density of a fluid (high expansion rates). If the fluid is under compression, the magnitude of the vorticity will increase. This can be easily seen from conservation of angular momentum principles. This is like the ice-skater spiraling around on one skate. As the skater arms are brought in close to the body, the rotation rate increases, whereas as the arms are extended, the rotation rate decreases. Note that this is not a generation or destruction term in the sense of creating or destroying vorticity. It acts to redistribute existing vorticity. In this course we will be dealing only with incompressible flows so that this term will not play a role. It is however, a very important mechanism in combustion and nonreacting compressible flows.

Baroclinic Torque The second term on the rhs is called the baroclinic torque. This term results in a generation of vorticity from unequal acceleration as a result of nonaligned density and pressure gradients. If the density gradient and pressure gradient are aligned, this term is zero. Say the pressure gradient is perpendicular to the density gradient (the case in which the baroclinic torque is largest. It is easy in this case to see how vorticity is generated. The lighter density fluid will be accelerated faster than the high density fluid, resulting in a shear layer, thus the generation of vorticity. Obviously, this term is only non zero in a variable density flow. variable density flows. Again, in reacting flows and compressible (reacting or nonreacting) flows where large density gradients can occur, this term can play a significant role in the development of the flow field. It is not necessary for the fluid to be compressible. In a variable density, but incompressible fluid like the atmosphere, this mechanism often affects the dynamics.

Viscous Diffusion The last term in Eq. 6.1 simply describes the effects of viscous diffusion on the vorticity distribution. As a result of viscosity, the vorticity in a flow tends to diffuse in space. In high Reynolds number turbulent flow, viscous diffusion of vorticity will be dominated by the other mechanisms in the vorticity transport equations. This will be the case unless the length scales of the turbulence are small enough where the contributions of viscosity can be important. The effects of viscosity on the large scale vortex structures in a turbulent flow are generally small. The development of the large scale features is therefore independent of viscosity. This realization is very important for the justification of “Direct Numerical Simulations” which we will discuss towards the end of this course.

Consider a constant density flow for a moment. If the flow is also two-dimensional, the vorticity equation reduces to

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \quad (6.3)$$

Under this restriction, the vorticity simply acts as a passive scalar that follows fluid particle paths (except for the influence of viscosity, which is small for high Reynolds number flow). The vorticity vector is confined to a plane perpendicular to the flow, and no enhancement of vorticity or transport to smaller scales by vortex stretching mechanisms is possible.

Vortex Stretching The term $(\boldsymbol{\omega} \cdot \nabla) \mathbf{V}$ is called the vortex stretching term and can be argued to be the most important mechanism in the turbulence dynamics. It represents the enhancement of vorticity by stretching and is the mechanism by which the turbulent energy is transferred to smaller scales. In two dimensional flow, this term is identically zero. Although two-dimensional flows can exhibit highly random character, they lack this distinctive mechanism and their

development is qualitatively different from three-dimensional flows. This is just another example of the difficulty that has been encountered in trying to give an operational definition of turbulence.

6.2 Qualitative Aspects of the Energy Cascade

The vortex stretching term is essential to the 3-D structure of turbulence and to the development and amplification of turbulence as we will see below.

In section 3 we discussed characteristic length scales associated with the kinetic energy and viscous dissipation. Most of the kinetic energy being associated with the largest structures (small wavenumbers), and the dissipation of this energy into heat occurring at the smallest scales (large wavenumbers). With this as the case, there must be a mechanism for the transfer of energy across these scales. This mechanism is “vortex stretching.” Let us look at this from two points of view.

Consider first an element of fluid in a turbulent flow having a vorticity vector ω . Denote the two ends of this element as 1 and 2. Since we are considering a turbulent flow, each end of this element will be subject to random (no preferred orientation) velocity perturbations. Making an analogy from the “random walk,” points one and two will tend to grow further and further apart with time. This increasing separation of the two ends of this element results in a stretching of the element (hence the term vortex stretching), and a decrease in the diameter of this element (length scale reduction). Conservation of angular momentum considerations now complete the picture.

The angular momentum is proportional to ωr^2 . Decreases in r therefore imply an increase in ω if angular momentum is conserved. The effects of vortex stretching are now seen to reduce the length scales of the turbulence in the two directions perpendicular to the turbulence, while intensifying the vorticity, yielding the transfer of energy to smaller scales.

Effects of vortex stretching on the kinetic energy of the turbulence can also be addressed with this simple model. As stated above, the angular momentum is conserved and is proportional to ωr^2 . The kinetic energy on the other hand, is proportional to $\omega^2 r^2$. If angular momentum is conserved as we reduce length scales, the kinetic energy must increase (at the expense of the kinetic energy that does the stretching). The source of this energy is larger scale motions (as creatively put by Richardson (section 1)). At the largest scales of the flow, there must be some external input energy mechanism, or else the total kinetic energy would eventually decay by the actions of viscosity. This energy source can be of many forms. In a boundary layer, or free-shear layer, the energy source is the mean shear (we will illustrate this by analyzing equations for the mean turbulent kinetic energy later in the course). It may also be due to heating. A heated plate will result in a decrease in density near the plate, which results in an unstable stratification and drives the flow. The energy source can take many other forms as well (magnetic, electrical, chemical, etc.).

Next, we would like to relate the qualitative discussion above to the vorticity trans-

port equation. In Cartesian tensor notation the vortex stretching term is expressed as

$$\omega_j \frac{\partial u_i}{\partial x_j} \quad (6.4)$$

If we decompose the deformation tensor that appears in Eq. into its symmetric (strain, S_{ij}) and antisymmetric (rotation, R_{ij}) components, we obtain:

$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j S_{ij} + \omega_j R_{ij} \quad (6.5)$$

By expanding out the antisymmetric component, it can be shown to be identically zero so that

$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j S_{ij} \quad (6.6)$$

The vortex stretching term is then seen to represent the amplification of vorticity by the local strain rate. A positive strain will result in an increase in the magnitude of vorticity, a negative strain gives a decrease. Similarly, straining motions aligned with a vortex element will result in a length scale reduction and intensification of vorticity as described above.

A Definition of Turbulence There are no general agreements on a definition of turbulence. We have already discussed that it is usually described in terms of unpredictability and randomness. An inherent aspect of turbulence, and what gives it much of its distinctive character, however, is the three-dimensional vortex stretching. With this as one of the defining characteristics, Bradshaw¹ has come up with the following definition: *Turbulence is a three-dimensional time-dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between a minimum determined by the viscous forces and a maximum determined by the boundary conditions of the flow. It is the usual state of fluid motion except at low Reynolds numbers.*

In the following subsection we will apply these ideas of vortex stretching to describe the dynamics of the flow in various wavenumber regimes.

6.3 Equilibrium Range Theories

In the previous section we stated that no real flow is completely isotropic, but alluded to the idea that there are regimes where the assumption of homogeneity and isotropy may be valid. We also discussed the idea that a wide range of temporal and spatial scales characterize turbulent flows. In each of these ranges the flow has different characteristics. At the largest scales (in the range of the integral scales),

¹In his book *An Introduction to Turbulence and Its Measurement*.

the fluctuations draw their energy directly from the mean motion. The turbulence is highly anisotropic, and its statistical behavior can vary significantly from flow to flow. The behavior of the flow at these large scales is strongly dependent on the geometry of the flow field, and the physical processes that the flow may be undergoing. At scales much smaller than this, but still larger than the dissipation lengths ($L \gg l_i \gg l$), the dynamics take on a more universal behavior independent of the large scale, anisotropic flow field. This is a result of the random, three-dimensional vortex stretching which reduces length and time scales. As we approach scales significantly smaller than the large energy containing eddies, the length and time scales are reduced to such an extent that they respond very rapidly relative to the large scale motions. As the energy cascades through the spectrum, directional preferences that the large scale motions exhibit are lost.

A way of looking at this is to visualize physically what effects the vortex stretching is producing in the flow. Assume we have a mean flow that is producing a stretching only in one direction, say z . This will intensify any motions in the x and y directions. The intensification of motion in the x direction will cause the same in the y and z directions, while the increased y motions cause higher levels of fluctuation in z and x . This process continues so that the fluctuations in x , y , and z eventually become isotropic at the small scales (if viscosity doesn't damp out the fluctuations before this occurs).

The smallest scales in the flow will exhibit the highest velocity gradients. (Small length scales and intense vorticity). As a result of the high gradients, kinetic energy is efficiently converted into heat (internal energy). This is the dissipation range. Over length scales smaller than this, no spatial variation can be maintained due to the dominating influence of viscosity which yields a hydrodynamically stable flow at these scales.

In the high wavenumber range (small length scales), the character of the turbulence can therefore be argued to be independent of the external flow conditions. Throughout this wavenumber range, the eddies obtain their energy by inertial transfer from larger eddies. There is a steady flux of energy across the spectrum, and the eddies are approximately in equilibrium with each other. This range of length scales is called the *equilibrium range*. In this range, there are only two parameters that determine the character of the turbulence, the dissipation and viscosity.

These ideas were first put forth by Kolmogorov[1], in his two famous hypotheses. These hypotheses, along with what they imply about the structure of turbulent flow, rank among the greatest contributions to turbulence theory.

Kolmogorov's 1st Similarity Hypothesis: At sufficiently high Reynolds numbers there is a range of high wavenumbers where the turbulence is statistically in equilibrium and uniquely determined by the parameters ϵ and ν . This state of equilibrium is universal.

The term universal is used here to emphasize that the character of the turbulence in this range does not depend on any specific mechanisms of the mean flow. The only

parameters that play any role in the description of the turbulence are the dissipation and viscosity. From this we can derive characteristic length, time and velocity scales of the small scales. The Kolmogorov length scale is defined using dimensional analysis as:

$$\eta = \left(\frac{\nu^3}{\epsilon} \right)^{1/4} \quad (6.7)$$

Using ν and ϵ to form a velocity scale gives:

$$v_\eta = (\nu\epsilon)^{1/4} \quad (6.8)$$

It is interesting to compute the Reynolds number based on these scales:

$$Re_\eta = \frac{\eta v_\eta}{\nu} = \nu^{-1} \left(\frac{\nu^3}{\epsilon} \right)^{1/4} (\nu\epsilon)^{1/4} = 1 \quad (6.9)$$

The value of $Re_\eta = 1$ is a constant independent of any flow parameters. This is not unexpected as at the small scales viscosity dominates and the relative Reynolds number will be very small. This Reynolds number is characteristic only of the strong viscous region (dissipation range) and is not characteristic of the turbulence throughout the equilibrium range.

For completeness, it is noted that a characteristic time scale can be defined:

$$\tau_\eta = \frac{\eta}{v_\eta} = \left(\frac{\nu}{\epsilon^2} \right)^{1/4} \quad (6.10)$$

If the Reynolds number of the flow is very high, it can be expected that there are values of l_i much larger than η , but still within the equilibrium range, where the dissipation is very small compared with the energy flux through this region. Kolmogorov, in his second hypothesis specifically addressed this regime where $L \gg l_i \gg \eta$, where L is the integral scale and η is the Kolmogorov scale. In terms of wavenumbers, this region corresponds to:

$$k_L \ll k_i \ll k_\eta \quad (6.11)$$

Kolmogorov's 2nd Similarity Hypothesis: If the Reynolds number is sufficiently large, there exists a range of wavenumbers, $k_L \ll k_i \ll k_\eta$, where the turbulence is independent of ν and is unambiguously defined by the value of the dissipation, ϵ .

In this range the inertial transfer of energy is the primary parameter characterizing the turbulence. Hence, this range (given by Eq. 6.11) is called the *inertial subrange*.

For another, qualitative way, to look at the ideas that may have led to the formalization of these hypotheses, consider a flow field consisting of large eddies whose size and development is determined by the geometry and forcing conditions of the flow. Imbedded within the large eddies, are smaller eddies which, as we have qualitatively

sketched out, acquire their energy from the large eddies. But because the small eddies have time scales much smaller than the large eddies (we will show this to be a factor of $Re^{1/2}$), the small scales respond rapidly to any attempt by the mean motion and large eddies to order their structure. At small enough length scales the dynamics will then be independent of the large scale motion, as Kolmogorov has formally hypothesized.

6.3.1 Energy Spectrum in the Equilibrium Range

Because the character of the turbulence in the equilibrium range depends only on a small number of parameters, it is possible to derive relations for the form of the energy spectrum. In the equilibrium range the turbulence depends only on the wavenumber k , the dissipation ϵ , and the viscosity, ν . Dimensional analysis then gives the following form for the energy spectrum in this range.

$$E(k) = g\left(\nu^{3/4}\epsilon^{-1/4}k\right)\epsilon^{2/3}k^{-5/3} \quad (6.12)$$

Within the inertial subrange (in the equilibrium range) the effects of viscosity go to zero. In this subrange the energy spectrum becomes:

$$E(k) = C\epsilon^{2/3}k^{-5/3} \quad (6.13)$$

where $C = g(0)$. Eq. 6.13 is called the *Kolmogorov* spectrum and C the Kolmogorov constant. C is a “universal” constant whose experimentally determined value lies in the range of 1.5-2.5. A number of famous experiments have verified the functional form given by Eq. 5.12. Most notable were the measurements of turbulence spectra in Knights Inlet, (a tidal channel in British Columbia) by Grant et al.[2].

In other regions of the flow, the form of the spectrum cannot be determined from dimensional arguments alone. However, a differential equation for the energy spectrum can be derived (e.g., Hinze[3], Chapter 3). With appropriate approximation and simplification the form of the energy spectrum has been computed in the various wavenumber regimes of the flow.

A Final Note

The random energy transfer (vortex stretching) throughout the wavenumber spectrum is seen to homogenize the small scale disturbances. In any real flow then, it seems reasonable to assume that there are regions of the flow that can be treated as isotropic and homogeneous. With this justification, the study of isotropic flow is not just a mathematical necessity, but an approximation that should reasonably describe the properties of certain turbulent flows.

References

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