2 GOVERNING EQUATIONS

For completeness we will take a brief moment to review the governing equations for a turbulent fluid. We will present them both in physical space coordinates and wavenumber (or Fourier) space. The equations describing the motion of a fluid are simply expressions for the conservation of mass, momentum ($F = ma$), and energy. These equations, based on the laws of Newtonian mechanics and Thermodynamics are generally accepted to provide an exact model of turbulent motion.

2.1 Conservation of Mass

Consider an arbitrary fixed volume in space. The rate at which mass accumulates in that volume is simply:

$$\text{Mass accumulation} = \iiint_v \frac{\partial \rho}{\partial t} dv \tag{2.1}$$

where v represents the arbitrary volume. This rate of accumulation must equal the net rate at which mass is flowing into this volume which can be expressed as:

$$-\iint_A \rho \mathbf{V} \cdot d\mathbf{A} \tag{2.2}$$

where A represents the surface area through which mass is flowing. The minus sign in front of the integral represents the notation that the area vector is normal to the surface element and points *outward*. $d\mathbf{A}$ is an elemental area vector, (dA_x, dA_y, dA_z). The **bold** characters indicate vector quantities. Note that we could also express $d\mathbf{A}$ as $\mathbf{n}dA$ where $\mathbf{n}$ is the unit normal (outward pointing) and dA is the magnitude of the element of area.

Equating Eq. 2.1 and 2.2 gives

$$\iiint_v \frac{\partial \rho}{\partial t} dv + \iint_A \rho \mathbf{V} \cdot d\mathbf{A} = 0 \tag{2.3}$$

Using Gauss' divergence theorem (see any text on vector analysis) the surface integral on the rhs of Eq. 2.3 can be expressed as a volume integral. The resulting equation is:

$$\iiint_v \frac{\partial \rho}{\partial t} dv + \iiint_v (\nabla \cdot \rho \mathbf{V})\, dv = 0 \tag{2.4}$$

or

$$\iiint_v \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V}\right] dv = 0 \tag{2.5}$$

Eq. 2.5 is valid for any arbitrary volume, v. This can only be true if the integrand is identically zero. Thus, the equation expressing the conservation of mass (Continuity equation) is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \tag{2.6}$$

For an incompressible flow this equation reduces to:

$$\nabla \cdot \mathbf{V} = 0 \tag{2.7}$$

Note that Eq. 2.7 does not imply that the density is uniform throughout the fluid, only that it is not compressible. As an example consider a mixture of helium and oxygen at constant temperature and where the Mach number is small. Although there can be mixing between the two fluid of different density, local volume of fluid elements is conserved.

2.2 Conservation of Momentum

The conservation equation for momentum is simply a mathematical expression of Newton's Second Law, $F = ma$, that is stated here for a fixed amount of mass. When formulated for an arbitrary control volume (allowing for momentum flux across the boundaries and deformation of the boundaries), use of Leibnitz integration rule or the Reynolds Transport Theorem, results in an expression that can be stated as: *The rate of change of momentum in an arbitrary volume of fluid is equal to the net flux of momentum through the surfaces of that volume plus the net forces acting on that volume.* The forces acting will consist both of surface forces and body forces.

Let us consider first the surface forces. The surface forces result from the existence of a stress tensor. For a Newtonian fluid this stress tensor can be expressed as:

$$\sigma_{ij} = -p\delta_{ij} + \mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \nabla \cdot \mathbf{V} \delta_{ij} \right] \tag{2.8}$$

For an incompressible flow the stress tensor simplifies to:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} \tag{2.9}$$

where S_{ij} is the strain rate tensor and is given by:

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{2.10}$$

The surface force acting on an element of surface dA in the x direction is

$$df_x = \sigma_x \cdot d\mathbf{A} \tag{2.11}$$

or in Einstein index notation (because it's less ambiguous when dealing with tensors):

$$df_i = \sigma_{ij} dA_j \tag{2.12}$$

where f_i represents the force in the i^{th} direction. Repeated indices represent summation over that index.

A number of body forces can be acting on the fluid. These may include gravity forces, Coriolis force (rotation) or electrical or magnetic forces for appropriate conducting fluids. Considering only the gravity force, we have for an element of volume:

$$df_i = g_i dv \tag{2.13}$$

Putting all this together and integrating over our arbitrary volume and surface of this volume gives:

$$\iiint_v \frac{\partial \rho V_i}{\partial t} dv + \iint_A (\rho V_i) V_j dA_j = \iiint_v g_i dv + \iint_A \sigma_{ij} dA_j \tag{2.14}$$

Again applying Gauss' divergence theorem to convert the surface integrals into volume integrals and noting that the expression must hold for any control surface, we arrive at the following equation for the conservation of momentum:

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x_j} = \rho g_i + \frac{\partial \sigma ij}{\partial x_j} \tag{2.15}$$

For an incompressible fluid with constant viscosity this equation reduces to:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \mathbf{g} \tag{2.16}$$

For an incompressible fluid, Eqs. 2.7 and 2.16 describe the evolution of the turbulent velocity field. With appropriate initial and boundary conditions, this set of equations could, in principle, be solved for the time development of the velocity field. In practice, however, there are some obvious difficulties with this. The set of equations given by 2.7 and 2.16 are a set of coupled, nonlinear, partial differential equations, with, in the most general cases, complex initial and boundary conditions. As a result, exact solutions exist only for very simplified conditions, all being for laminar flow. To solve these equations for problems of practical combustion applications, numerical techniques must be used. As we will see later, there are also fundamental difficulties with the numerical solution. Approximate methods for the numerical solution of these equations will be discussed in subsequent lectures.

For the moment let us make some observations about turbulence based on equation Eq. 2.15 (ignore gravity for the moment). The only source of nonlinearity in this equation is the convective term. If we ignore nonlinear convection, Eq 2.15 is a linear equation that simply describes the viscous damping of the fluid. The nonlinear convective term thus provides the mechanism for feeding energy to the various length scales. Turbulence is a nonlinear process, and we must attempt to understand and describe these nonlinear interactions if we are to have any hope of dealing with turbulence.

2.2.1 Fluid Deformation

In the first section of the notes the velocity gradient tensor, $\frac{\partial u_i}{\partial x_j}$, was introduced to illustrate the decomposition of any tensor into its symmetric and antisymmetric components. Repeating that decomposition here:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = S_{ij} + R_{ij} \tag{2.17}$$

S_{ij} represents a pure straining motion and is called the *rate of strain* tensor. The second term, R_{ij}, describes a rigid body rotation. The deformation of a fluid element can therefore be separated into these two distinctly different contributions. Details on the nature of these two mechanisms of fluid deformation can be found in most graduate level fluid mechanics textbooks, e.g.,Panton[1].

2.3 Fourier Transforms

Insight into the behavior and structure of turbulent flow is facilitated by considering several different viewpoints from which to study it. One convenient space is frequency or wavenumber space. One approach is to start a theoretical study of turbulence by transforming the governing equations (Eqs. 2.6, 2.15) to their counterparts in wavenumber, or Fourier space. One thing accomplished by this transformation is that differential operators are turned into algebraic multipliers. From some perspectives, this results in a simplification in the analysis of the governing equations.

Furthermore, it is often more natural to treat many of the turbulence phenomena in terms of frequencies and wavenumbers. This includes important dynamical properties such as the kinetic energy and energy dissipation distribution (spectra). For example, the energy spectra, $E(k)$, yields the energy distribution in the flow as a function of wavenumber, where small wavenumbers represent the large eddies, and large wavenumbers represent the small eddies. The Fourier transform decomposes the velocity (or any dependent variable) field into its component waves of different wavelengths. Understanding the interaction of information in the frequency or wavenumber domain often yields useful information in understanding the physical domain.

The general definition of the Fourier Transform of a function, $f(t)$ is

$$g(\omega) = \Im\{f(t)\} \equiv \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \tag{2.18}$$

and the inverse transform is

$$f(t) = \Im^{-1}\{g(\omega)\} = 2\pi \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}\omega \tag{2.19}$$

The above describes the transform from a time domain to a frequency space. Instead of looking at the transform from time to frequency, the Fourier Transform can be defined in terms of a transform from physical space domain to a wavenumber domain:

$$g(\mathbf{k}) = \Im\{f(\mathbf{x})\} \equiv \int_{-\infty}^{\infty} f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} \tag{2.20}$$

and the inverse transform is

$$f(\mathbf{x}) = \Im^{-1}\{g(\mathbf{k})\} = 2\pi \int_{-\infty}^{\infty} f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k} \tag{2.21}$$

2.3.1 Discrete Fourier Transform

Consider a velocity component in a turbulent flow. This variable is in general a function of both space and time. For a cubic domain with sides of length L, with periodic boundary conditions the discrete Fourier transform (space to wavenumber transformation) is defined as:

$$\mathbf{V}(\mathbf{x}, t) = \left(\frac{2\pi}{L}\right)^3 \sum_{k_1,k_2,k_3=-\infty}^{\infty} \hat{\mathbf{V}}(\mathbf{k}, t) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \tag{2.22}$$

The inverse transform is given by:

$$\hat{\mathbf{V}}(\mathbf{k}, t) = \sum_{x_1,x_2,x_3=-\infty}^{\infty} \mathbf{V}(\mathbf{x}, t) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) \tag{2.23}$$

For a general flow where $L \to \infty$, the integral Fourier transform is defined as above:

$$\hat{\mathbf{V}}(\mathbf{x}, t) = \iiint \mathbf{V}(\mathbf{x}, t) \exp(-i\mathbf{k}\cdot\mathbf{x}) d\mathbf{x} \tag{2.24}$$

The derivative of Eq. 2.22 is simply obtained by differentiating the series expansion term by term:

$$\frac{\partial \mathbf{V}(\mathbf{x}, t)}{\partial x_1} = \left(\frac{2\pi}{L}\right)^3 \sum_{k_1,k_2,k_3=-\infty}^{\infty} ik_1 \hat{\mathbf{V}}(\mathbf{k}, t) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \tag{2.25}$$

or

$$FT\left(\frac{\partial U(x)}{\partial x}\right) = ik\hat{U}(k) \tag{2.26}$$

where FT indicates the Fourier Transform.

2.3.2 Fourier Transform of Governing Equations

The transforms defined above can be applied to the governing equations to give a set of coupled algebraic equations. The transforms of the linear terms are straight forward. The treatment of the nonlinear terms is slightly complicated. To obtain the Fourier transform of the nonlinear convective term, consider a multiplication of the type $W = U_i U_j$ (we will drop the explicit t dependence in the following equations; it will be implicitly assumed):

$$
\begin{aligned}
W(x) &= \sum_{\|k\|\leq\infty} w(k)\exp(ikx) \\
&= U_iU_j = \left[\sum_{\|k\|\leq\infty} \hat{U}_i(k)\exp(ikx)\right]\left[\sum_{\|k\|\leq K} \hat{U}_j(k)\exp(ikx)\right]
\end{aligned}
\qquad (2.27)
$$

In Eq. 2.27 w_k will then be given by the convolution sum:

$$
w_k = \sum_{p+q=k,\|k\|\leq\infty} \hat{U}_i(p)\hat{U}_j(q) \qquad (2.28)
$$

Now Fourier transforming Eqs. 2.7 and 2.16 then gives the following:

$$
\mathbf{k}\cdot\hat{\mathbf{V}}(\mathbf{k}) = 0 \qquad (2.29)
$$

$$
\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{V}_i(\mathbf{k}) = -i\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_i(\mathbf{k_2}) - ik_i\hat{p}(\mathbf{k}) \qquad (2.30)
$$

At this point a few things can be pointed out from Eqs. 2.29 and 2.30. First, the nonlinear terms give rise to interactions among a triad a wave numbers k, k_1, and k_2 such that $k_1 + k_2 = k$. From Eq. 2.29 it is seen that $\hat{V}(k)$ must lie in a plane perpendicular to $\mathbf{k}$ since the dot product is zero. ($\mathbf{A}\cdot\mathbf{B}$ is identically zero if the vector $\mathbf{A}$ is perpendicular to $\mathbf{B}$) The pressure gradient term $ik_ip(\mathbf{k})$ is parallel to $\mathbf{k}$ since p is a scalar. Pressure can be eliminated from Eq. 2.30 by taking the dot product of 2.30 with $\mathbf{k}$. Using the incompressibility condition 2.29, the left hand side of 2.30 will be zero (after taking the dot product). We are then left with:

$$
-ik_i\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_i(\mathbf{k_2}) = ik_ik_i\hat{p}(\mathbf{k}) \qquad (2.31)
$$

Since $k_ik_i = k^2$ (the magnitude of the wave number), we can write the Fourier transform of the pressure as:

$$
\hat{p}(\mathbf{k}) = -\frac{k_\alpha}{k^2}\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_\alpha(\mathbf{k_2}) \qquad (2.32)
$$

Note that we have change the repeated index i to α. This is O.K since it is just a dummy. Now using Eq. 2.32 in 2.30 gives

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \nu k^2\right)V_i(\mathbf{k}) &= -i\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_i(\mathbf{k_2}) \\
&\quad +ik_i\frac{k_\alpha}{k^2}\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_\alpha(\mathbf{k_2})
\end{aligned}
\qquad (2.33)
$$

In most of the literature Eq. 2.33 is usually written in the more compact form:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{V}_i(\mathbf{k}) = -ik_\alpha P_{ij} \sum_{k_1+k_2=k} \hat{V}_j(\mathbf{k_1})\hat{V}_\alpha(\mathbf{k_2}) \tag{2.34}$$

where

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{2.35}$$

The transformed equations are now a set of algebraic equations for the coefficients $\hat{V}(k)$. Starting with prescribed initial conditions, these equations can be integrated in time. As we will see, these equations cannot be solved exactly (due to computer limitations) for general high Reynolds number flow. Although the range of wavenumbers k is finite due to viscosity (the sum in Eq. 2.34 is over finite k), we will see in the next section that the magnitude of the largest wavenumber increases as the Reynolds number increases. Many recent approaches to turbulence modeling attempt to solve Eq. 2.34 exactly only for a small range of k, and model the interactions with the higher wavenumbers. This is the idea of *Large Eddy Simulation* (keep in mind that small k corresponds to large eddies). For now we leave this discussion here.

References

[1] R. L. Panton. *Incompressible Flow, 2nd Edition.* Wiley-Interscience, New York, 1996.