Linear Eddy Mixing Model

Motivation:

Incorporate distinct influences of turbulent stirring and molecular diffusion at all scales of motion.

No models thus far have made this distinction

Want to capture effects of Sc (CD, IEM can't do this)

To do this, one must resolve all scales of motion

Not feasible in multi-D flow

So, limit description of model to one spatial dimension

1-D convection-diffusion equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial u \phi}{\partial x} = D \frac{\partial^2 \phi}{\partial x^2} + \dot{\omega}_\phi$$

if we consider reaction too

Q? What can the one spatial dimension represent with respect to problems of practical interest?

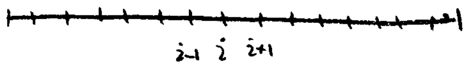
How can you have any type of interesting convection in 1-D?

Before we answer this let's look at how the model treats the various physical processes.

Diffusion:

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}$$

In 1-D (where the model is defined) we can simply solve the 1-D diffusion equation numerically



$$\phi_i^{n+1} = \phi_i^n + \Delta t \, D \left[\frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{\Delta x^2} \right] \qquad \bigstar$$

ϕ_i^n — n ← temporal index; i ← spatial index

Simple 1st order time, 2nd order space finite difference solution

(Can use whatever scheme you like)

The point is, we treat diffusion by a deterministic numerical solution of the diffusion equation of a resolved scalar field

Δx = grid point spacing

Δt = time increment

Can also explicitly treat reaction:

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} + \dot{\omega}_\phi$$

just add $\Delta t \, \dot{\omega}_\phi$ to r.h.s. of $\bigstar$.

The assumption is all scalar fields are resolved so reaction is handled exactly

B) Turbulent Convection

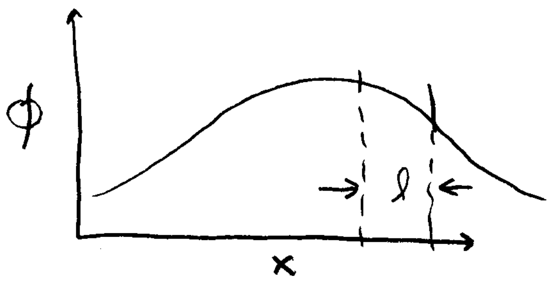
Key to model

Turbulent convection treated by random rearrangements (subject to rules) of the scalar field along a line. The rules governing this process are determined so that when carried out, give statistics representative of real turbulent flows

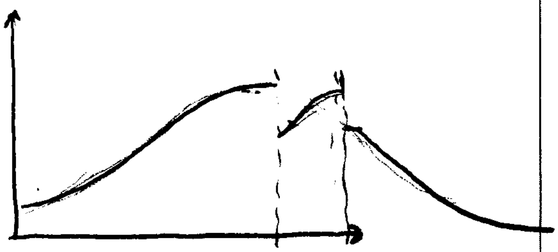
Simplest example "Block Inversion"

Select a region of the domain

"Invert" around its center



original field



Scalar field after inversion

Can make analogy with action of "eddy" - Although better to describe in terms of statistical effects

We'll show a better process later.

Process is characterized by

$\lambda \rightarrow$ rate paramenter (how often do events occur) in general $\lambda = f(x,t)$

$f(l) \rightarrow$ length scale distribution

(need to select size of region for rearrangement)

So linear eddy model involves

1) Defining a meaningful 1-D domain for application program

2) Continuous solution of diffusion equation in this domain...

3) ...interupted by rearrangement events

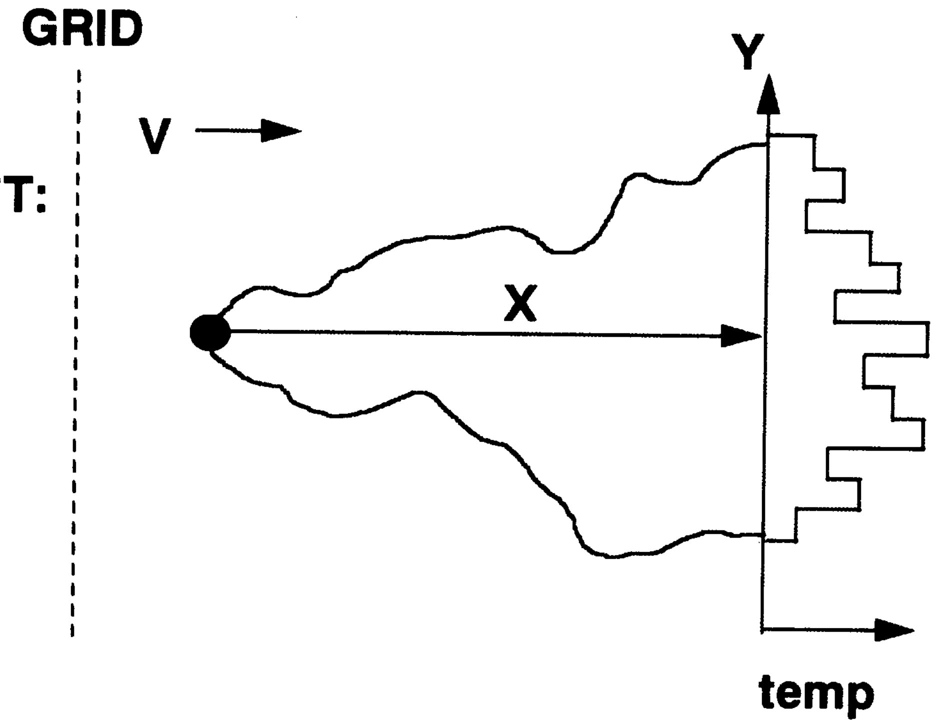
λ, $f(l)$ are specified as velocity field statistics are assumed known

Before specifying λ, $f(l)$ let's look at a couple of examples where LEM has been used.

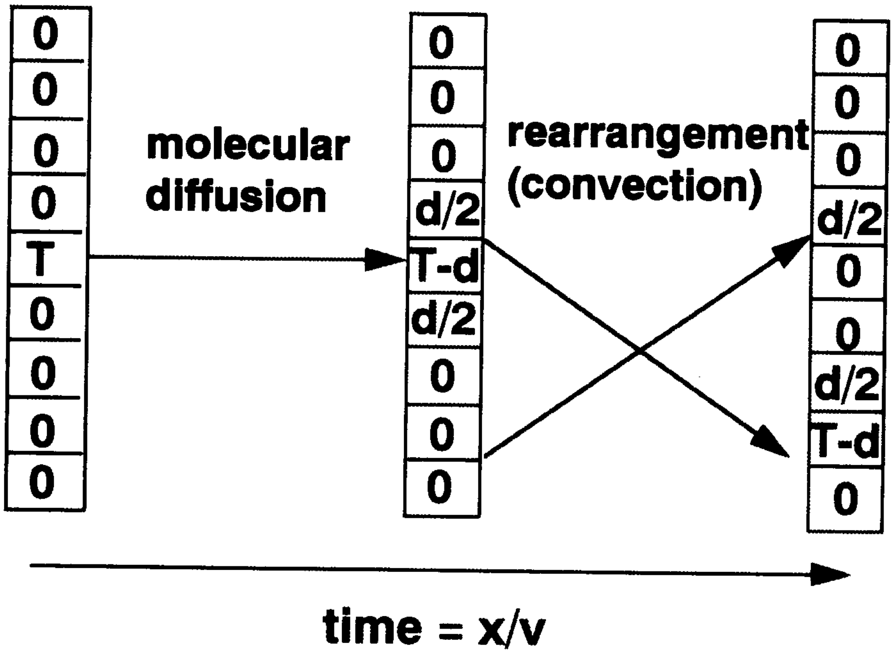
LEM5

APPLICATION TO SCALAR TRANSPORT

EXPERIMENT:



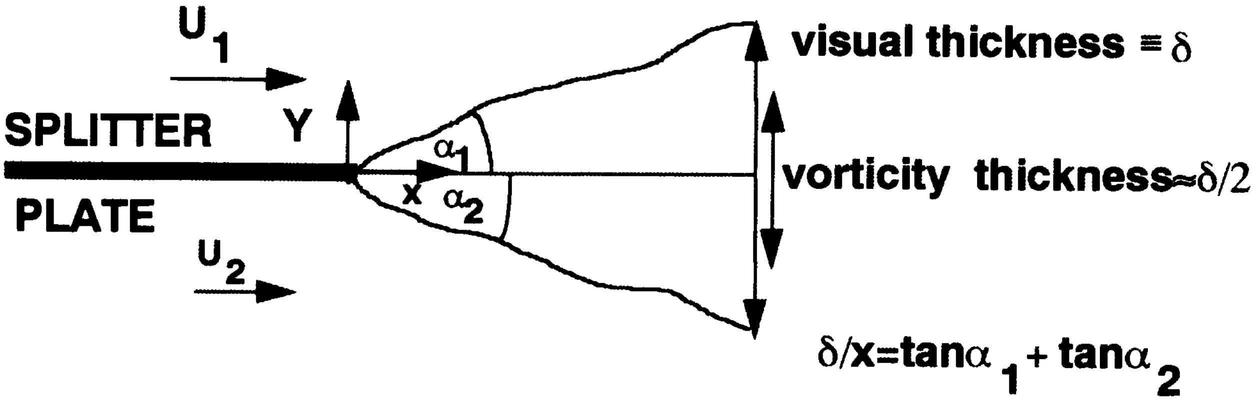
MODEL:



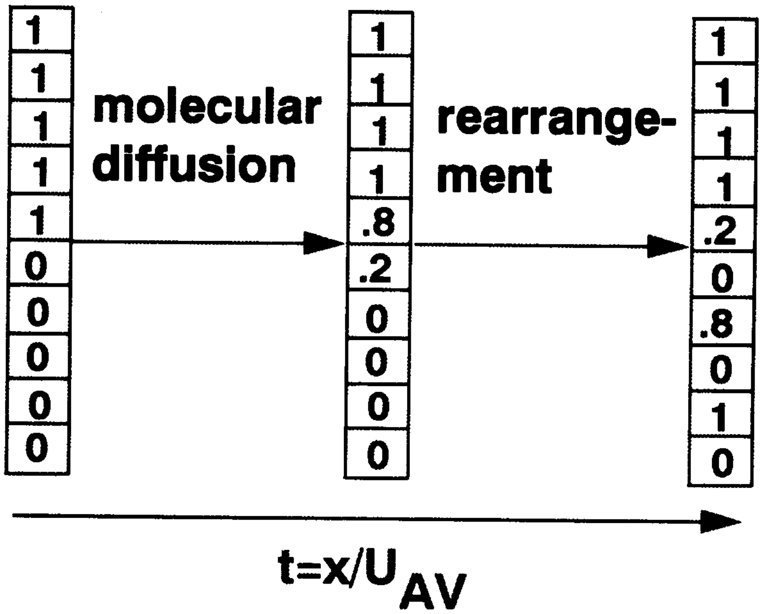
LEM6

APPLICATION TO SHEAR LAYER MIXING

EXPERIMENT:



MODEL:

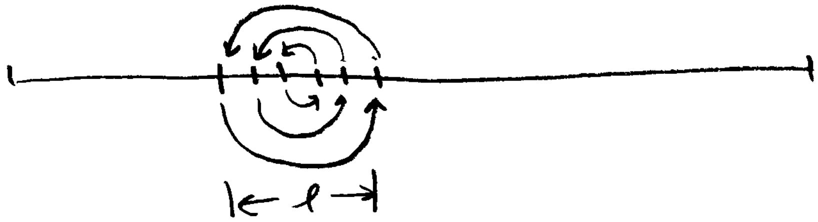


$U_{AV} = (U_1 + U_2)/2$

Parametrization of the stirring (rearrangement events)

λ (rate parameter) calculation

Observation: Rearrangement events result in random walk of marker particle along the line



There is a diffusivity associated with the random walk (See earlier notes)

Random walk diffusivity

$$D_T = \frac{1}{2} N \langle x^2 \rangle \quad \star$$

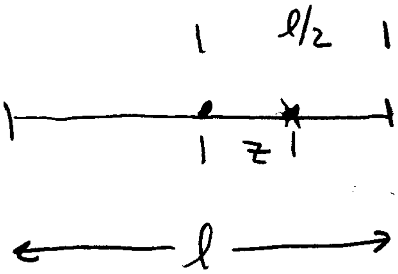
N = frequency of events

$\langle x^2 \rangle$ = mean square displacement per event

The key to what we will do is equating the random walk diffusivity to the turbulent diffusivity! $\star\star$

But for now, consider a particle displaced by an eddy of size l

Any particular marker particle lies within a distance $l/2$ of the center



Particle is a distance z from center

Frequency of event, $N = \lambda \ell$

$$\lambda = \frac{1}{m\text{-sec}}$$

For an event, displacement of particle is $2z$ and particle can be located anywhere in "eddy", so z is uniformly distributed in $[-\frac{\ell}{2}, \frac{\ell}{2}]$

$$\Rightarrow \langle x^2 \rangle = \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} (2z)^2 dz = \frac{\ell^2}{3}$$

This is for the "block inversion" stirring event.

But turbulent flows contain a wide range of length scales, given by some distribution, $f(\ell)$

Diffusivity due to eddies in range $[\ell, \ell + d\ell]$:

$$D_t(\ell) = \frac{\lambda}{6} \ell^3 f(\ell) d\ell$$

$$\left(N = \lambda \ell, \ \langle x^2 \rangle = \frac{\ell^2}{3} \right)$$ ← see ✱, LEM 7

Total diffusivity caused by all eddies up to size ℓ

$$D_t(\ell) = \int_{\eta}^{\ell} \frac{\lambda}{6} \ell^3 f(\ell) d\ell \quad ✱$$

But still need to specify λ & $f(\ell)$

To do this we now bring in some turbulence scalings

$$D_t(L) \sim UL = \nu Re_L \quad \& \quad D_t(\ell) \sim \nu Re_\ell$$

U characteristic velocity

If we say $Re_\ell \sim \left(\frac{\ell}{\eta}\right)^{4/3}$ then we can say

ℓ dependence of $\bigstar$, pg. LEM8 scales as $\left(\frac{\ell}{\eta}\right)^{4/3}$

or $$\int \ell^3 f(\ell)\, d\ell \sim \left(\frac{\ell}{\eta}\right)^{4/3}$$

$$\sim \ell^{4/3}$$

$$\Rightarrow \underline{\underline{f(\ell) = c\,\ell^{-8/3}}}$$ ← gives proper ℓ-dependence $\bigstar$ for $f(\ell)$

The constant c is obtained by recognizing that

$$\int_\eta^L f(\ell)\, d\ell = 1 \qquad \bigstar\bigstar$$

Substitute $\bigstar$ in $\bigstar\bigstar$ and solve for c

gives $$C = \frac{5}{3} \frac{1}{\eta^{-5/3} - L^{-5/3}}$$

$$\& \quad f(\ell) = \frac{5}{3} \frac{\ell^{-8/3}}{\eta^{-5/3} - L^{-8/3}}$$

From this we can solve for λ:

$$D_T(L) \sim \nu Re_L = \int_\eta^L \frac{\lambda}{6} \frac{5}{3} \frac{\ell^{-8/3}}{\eta^{-5/3} - L^{-5/3}} \, d\ell$$

$$= \frac{5}{24} \lambda \frac{L^{4/3} - \eta^{4/3}}{\eta^{-5/3} - L^{-5/3}}$$

Solve for λ:

$$\lambda = \frac{24}{5} \frac{\nu Re_L}{L^3} \frac{\left(\frac{L}{\eta}\right)^{5/3} - 1}{1 - \left(\frac{\eta}{L}\right)^{4/3}}$$

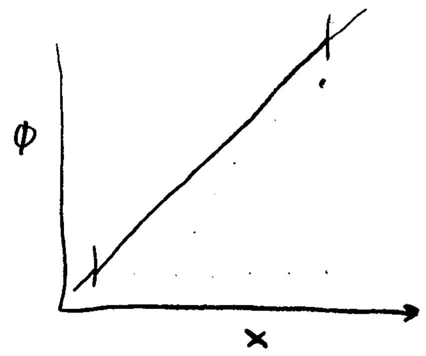
For high Re flows, $L \gg \eta$ The above can then be approximated by

$$\lambda = \frac{24}{5} \frac{\nu Re_L}{L^3} \left(\frac{L}{\eta}\right)^{5/3}$$

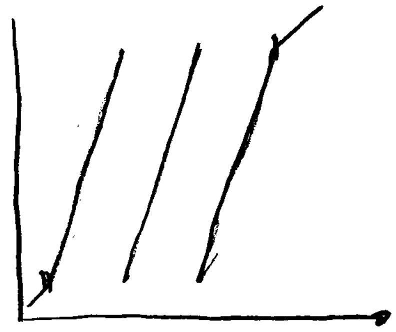
Other Mapping Rearrangements:

Triplet Map

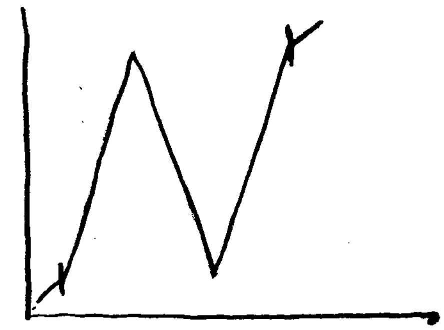
Consider a uniform gradient:



make 3 compressed copies →



invert center copy →



Properties: Does not introduce discontinuities

Increases scalar gradients

Increases level crossings (surface area increase)

$$\langle x^2 \rangle = \frac{4}{27} \ell^2$$

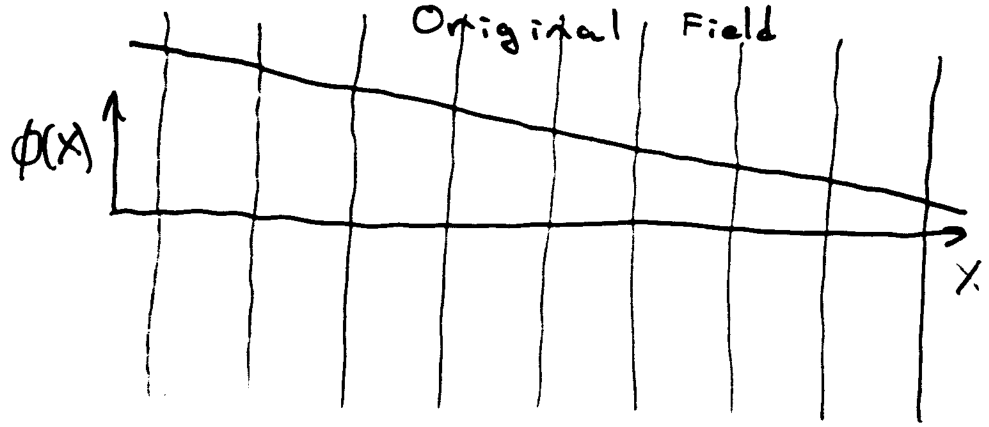
gives $\lambda = \frac{54}{5} \frac{1}{L\tau} Re^{5/4}$

($D_T = L^2/\tau = UL$ → so can write in terms of L, τ instead of L, ν, Re if we want

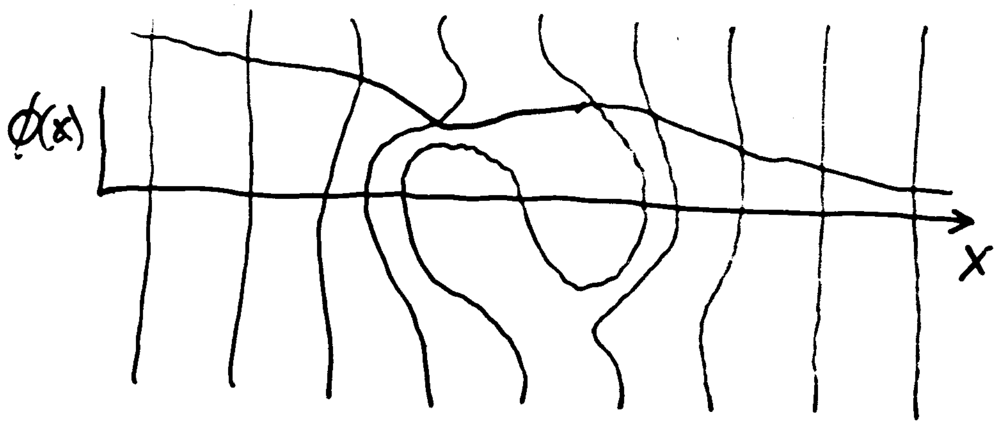
ANALOGY WITH TURBULENT EDDY

Effect of "clockwise eddy" on scalar gradient:

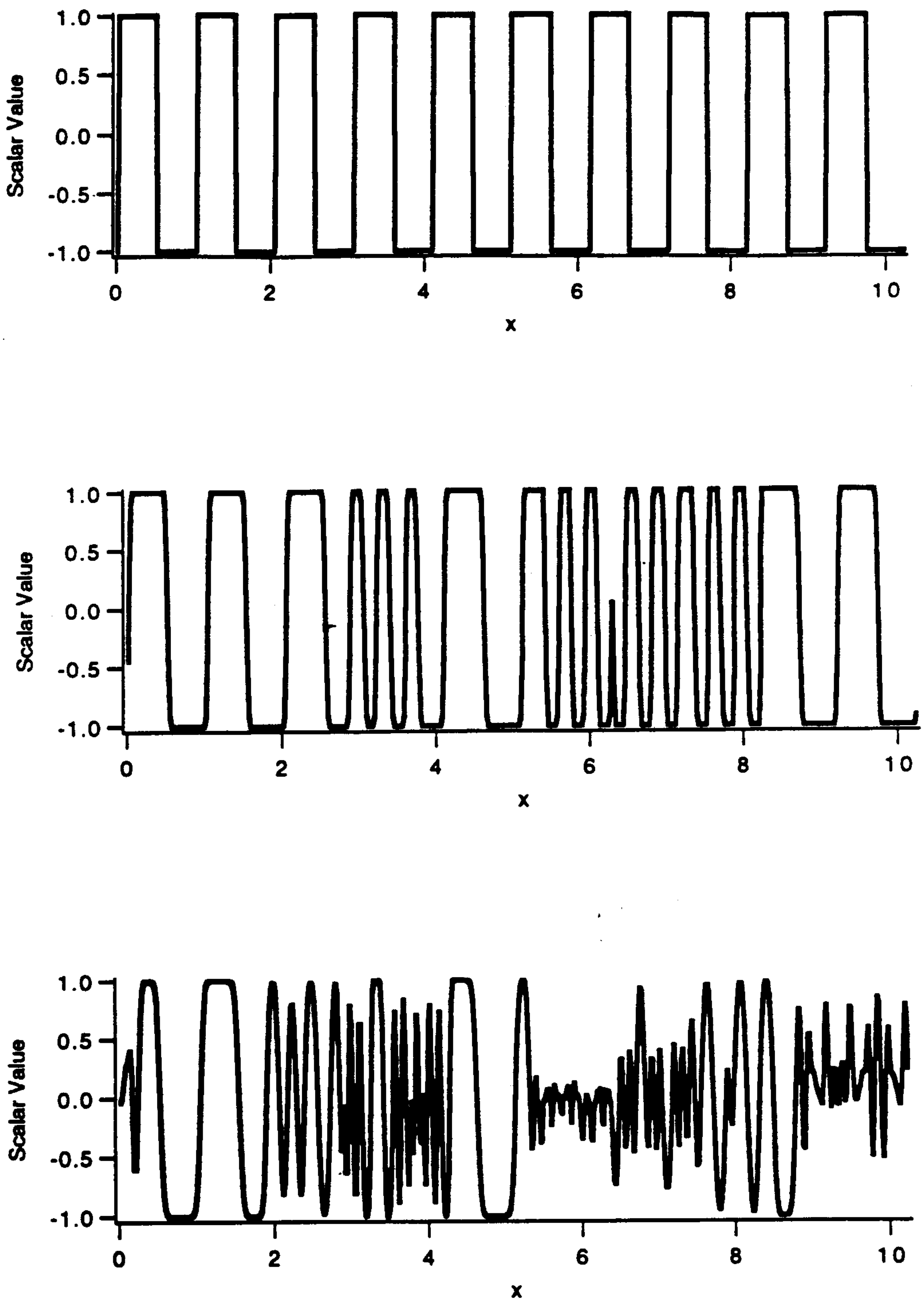
Original Field



After action of eddy



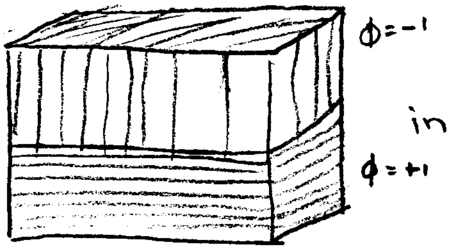
Linear Eddy Mixing Schematic



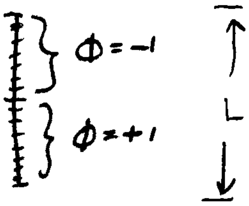
Linear Eddy Algorithm

Pick Configuration and define L

E.g., mixing in homogeneous turbulent flow



in 1-D



Say 6 gridpoints to resolve η. (for triplet map)

So for any given Re, need $6 \times Re^{3/4}$ grid points for full resolution (assuming one L in domain)

Simulation involves solving

Diffusion $\quad \frac{d\phi}{dt} = D \frac{\partial^2 \phi}{\partial x^2} \Rightarrow \phi^{n+1} = \phi^n + \frac{\Delta t_d D}{(\Delta x)^2}\left(\phi^{n-1} - 2\phi^n + \phi^{n+1}\right)$

This occurs continuously $\qquad \Delta t_d$ = time step for diffusion

Turbulent Stirring: Rearrangement events occur at a rate $R = \lambda L$ or $\Delta t_I = \frac{1}{\lambda L}$

so at every Δt_I, a rearrangement takes place.

This involves:

1) Selecting location (randomly if $\lambda \neq f(x)$)

2) Select l from $f(l)$ (random choice of l such that selected l's have pdf $f(l)$)

3) Carry out triplet map

How to generate a random # given a specified pdf

Given pdf $P_\phi(\psi)$

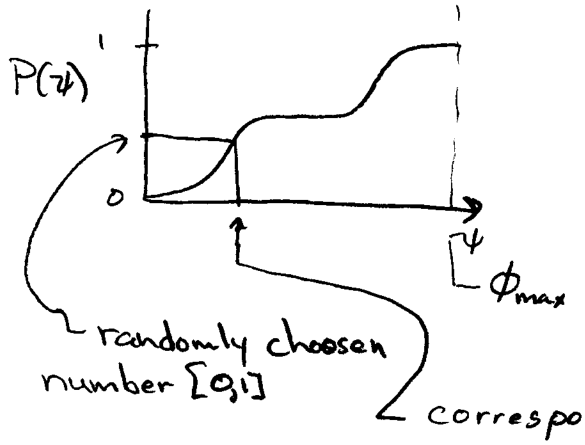
Form Cumulative $\mathbb{P}(\psi) = \int_{-\infty}^{\psi} p \, d\psi$

$\mathbb{P}(\psi)$ is monotonically increasing between 0 & $\mathbb{P}_{max}$

If $p(\psi)$ is such that $\int_{-\infty}^{\infty} p(\psi) d\psi = 1$, then

$$0 \leq \mathbb{P}(\psi) \leq 1$$

If $\mathbb{P}(\psi)$ can be generated analytically, solve for ϕ in terms of $\mathbb{P}(\psi)$



In LEM we get ℓ from CDF of $f(\ell)$

$$F(\ell) = \int_{\eta}^{\ell} f(\ell) \, d\ell$$

$$= \frac{1}{L^{-5/3} - \eta^{5/3}} \left(\ell^{-5/3} - \eta^{-5/3} \right)$$

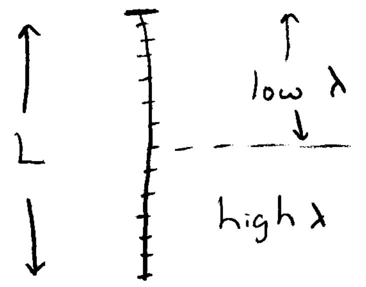
Solve for ℓ:

$$\ell = \left[F(\ell) \left(L^{-5/3} - \eta^{-5/3} \right) + \eta^{-5/3} \right]^{-3/5} \quad ☆$$

To obtain an ℓ, pick a random # uniformly distributed between 0 & 1 & substitute for $F(\ell)$. Then solve for L

For each inversion, select a new random #. This will give a set of random #'s for ℓ with the correct distribution, $f(\ell)$.

What if λ is not a constant



& L may change

Overall Rearrangement frequency given by $\int_0^L \lambda \, dx = R$

What about location? We know there is a higher probability for event to occur in region of high λ

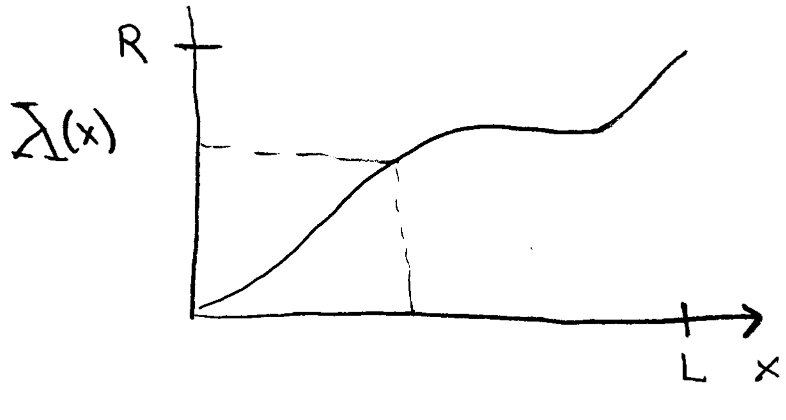
Say $\lambda = f(x)$

To find location, form

$$\Lambda(x) = \int_0^x \lambda(x)\,dx$$

$$\Lambda(0) = 0 \quad , \quad \Lambda(L) = R$$

Pick a random # between 0 & R, find corresponding x

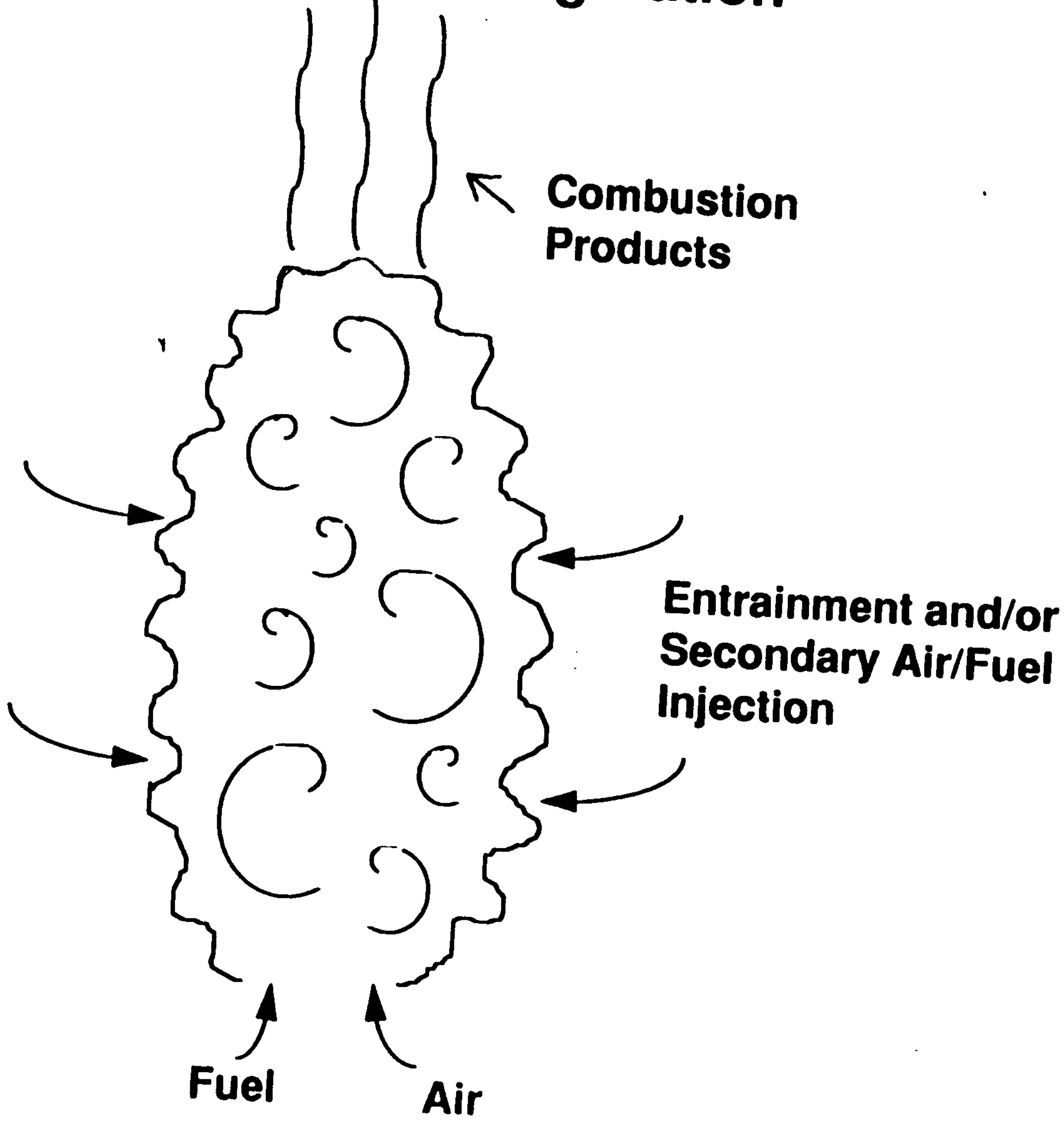


May have to do numerically

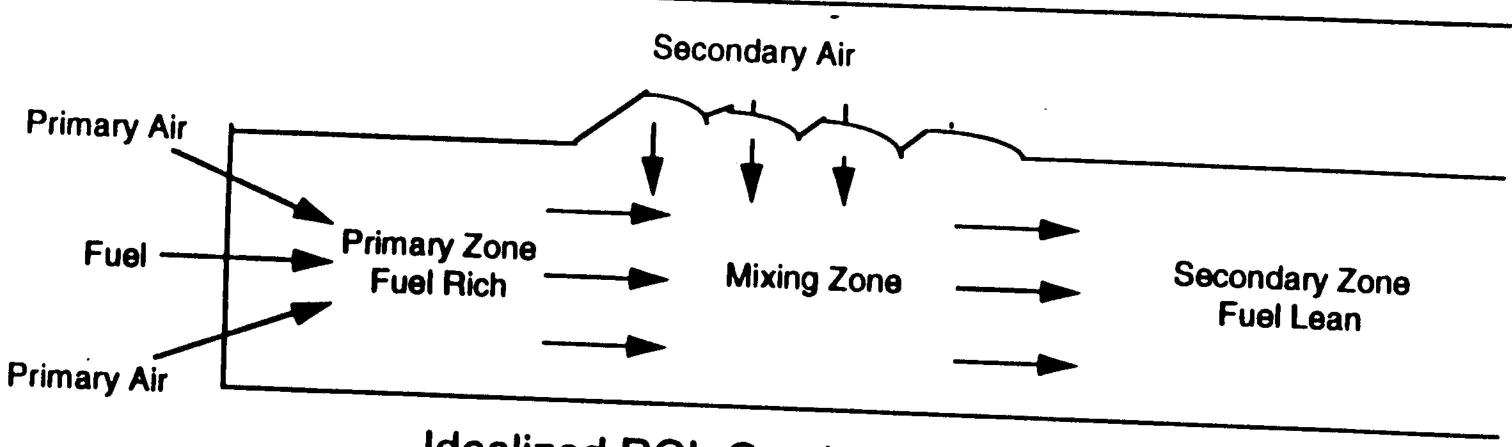
Generic Combustion Configuration

Physical Process

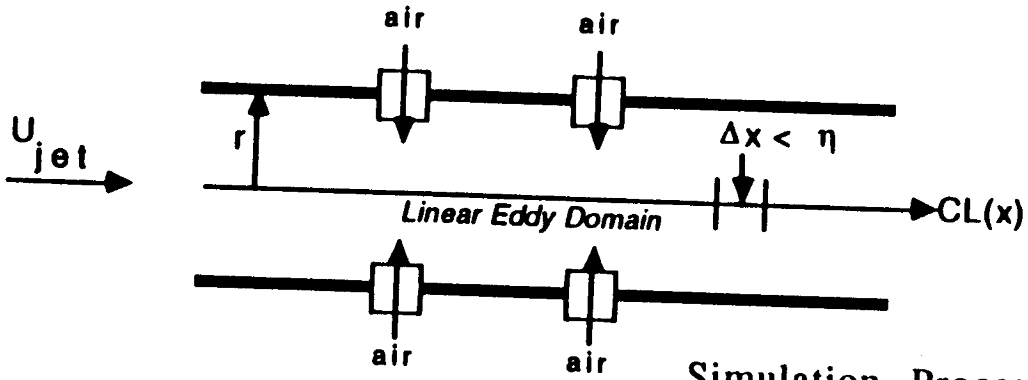
- Fuel/Air Feed
- Turbulent Stirring
- Molecular Diffusion
- Chemical Kinetics
- Heat Release/
  Thermal Expansion



LEM #6



Idealized RQL Combustor



Simulation Process Involves:

1) inlet feed
2) linear eddy mixing and reaction kinetics
3) secondary injection
4) downstream displacement

Summary of Linear Eddy Model

Explicit representation of turbulent stirring & molecular diffusion

Affordable high spatial resolution (1-D)

Computational efficiency allows for detailed parametric studies

Geometry features incorporated by configuration specific inputs

Good tool to examine physics of mixing

Limitations

Mixing Model → velocity statistics assumed known

Limited geometric applications (due to 1-D)

Limited application in engineering applications.