Derivation of equation for $\overline{u_i' u_j'}$

$$u_j' \frac{\partial u_i}{\partial t} = \ldots\ldots \quad \leftarrow \text{r.h.s N.S.} * u_j'$$

$$u_i' \frac{\partial u_j}{\partial t} = \ldots\ldots$$

add together

$$u_j' \frac{\partial u_i}{\partial t} + u_i' \frac{\partial u_j}{\partial t} = \ldots\ldots$$

$$\overline{u_j' \frac{\partial(\overline{u}_i + u_i')}{\partial t}} + \overline{u_i' \frac{\partial(\overline{u}_j + u_j')}{\partial t}} = \overline{\ldots\ldots}$$

$$\overline{u_j' \frac{\partial u_i'}{\partial t}} + \overline{u_i' \frac{\partial u_j'}{\partial t}} = \overline{\ldots\ldots}$$

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} = \overline{\ldots\ldots}$$

Let $\overline{u_i' u_j'} = R_{ij}$ The final full equation is:

$$\frac{\partial}{\partial t} R_{ij} + u_k \frac{\partial}{\partial x_k} R_{ij} = P_{ij} + T_{ij} - D_{ij} - \frac{\partial}{\partial x_k} J_{ijk}$$

$$P_{ij} = -\left(R_{ik} \frac{\partial \overline{u}_j}{\partial x_k} + R_{jk} \frac{\partial u_i}{\partial x_k} \right) \leftarrow \text{Production, closed}$$

$$J_{ijk} = -\nu \frac{\partial}{\partial x_k} R_{ij} + \overline{u_i' u_j' u_k'} + \frac{1}{\rho}\left(\overline{u_j' p'}\, \delta_{ik} + \overline{u_i' p'}\, \delta_{jk} \right)$$

$$D_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}$$

$$T_{ij} = \frac{1}{2} \overline{p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)}$$

J_{ijk} – Transport, requires modeling

$\frac{\partial}{\partial x_k} \overline{u_i' u_j' u_k'}$ ← turbulent transport of $\overline{u_i' u_j'}$

$\overline{u_j' p' \delta_{ik} + u_i' p' \delta_{jk}}$ – Not really know how this part behaves

DNS suggests it is small. So either ignore or absorb into $u_i' u_j' u_k'$

Modeling of $\overline{u_i' u_j' u_k'}$

Simplest $\overline{u_i' u_j' u_k'} \sim \frac{\partial \overline{u_i' u_j'}}{\partial x_k}$ ← gradiant transport

but $\overline{u_i' u_j' u_k'}$ is "rotationally invarient" (symmetric in all indices)

So take form:

$$C_{ijk} = \frac{2}{3} C \underset{\substack{\uparrow \\ \nu_t}}{\frac{k^2}{\varepsilon}} \left(\frac{\partial \overline{u_j' u_k'}}{\partial x_i} + \frac{\partial \overline{u_i' u_k'}}{\partial x_j} + \frac{\partial \overline{u_i' u_j'}}{\partial x_k} \right)$$

(Not only model – variations exist)

Dissipation:

$$D_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}$$

Assuming D_{ij} is a small scale process,

D_{ij} should be isotropic

$$D_{ij} = \frac{2}{3}\varepsilon \delta_{ij}$$

$$\varepsilon = \nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \Big\} \text{ obtained from } \varepsilon \text{ equation}$$

Not isotropic near walls - add damping & nonisotropy

Pressure Strain

The pressure & velocity correlations that appear in the Reynolds stress equation can be decomposed in several ways. As done here (which is common), we are left with a "pressure-strain term";

$$\frac{1}{2} p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right) = T_{ij}$$

This must be modeled. What to do?

To gain some understanding, an equation for the fluctuating pressure can be derived.

Before doing this, note $T_{ii} = 0$ (for incomp. flow) so it does not contribute to kinetic energy. It redistrubes energy between various Reynolds stress components (See Pope, pg 389)

To derive an equation for p', note the following:

Take full N.S. equation:

$$\frac{D\,\overline{u_i' + \overline{u}_i}}{Dt} = \text{----}$$

& Subtract mean momentum:

For an incompressible flow, this gives

$$\frac{Du_i'}{Dt} + u_j' \frac{\partial \overline{u}_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} \quad \star$$

Next, take $\nabla \cdot \star$

This gives

$$\nabla^2 \frac{p'}{\rho} = -\frac{\partial^2 (u_j' u_m' - \overline{u_j' u_m'})}{\partial x_j \partial x_m} - 2 \frac{\partial \overline{u}_j}{\partial x_m} \frac{\partial u_m'}{\partial x_j}$$

This can be solved by integrating & applying Greens Theorem (See pg 19 in Pope)

Solution is:

$$\frac{p(\vec{x})}{\rho} = \frac{1}{4\pi} \int_{Vol.} \left\{ \frac{\partial^2 (u_k' u_m' - \overline{u_k' u_m'})}{\partial x_k \partial x_m} + 2 \frac{\partial \overline{u}_k}{\partial x_m} \frac{\partial u_m'}{\partial x_k} \right\} \frac{dV}{\vec{r}}$$

To get the pressure-strain correlation, multiply the above by $\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}$ & average

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}\Bigg|_{\vec{x},t} = \frac{1}{4\pi}\int_{Vol} \left[\overline{\left(\frac{\partial^2 u_k' u_m'}{\partial x_k \partial x_\ell}\right)_{\vec{r},t} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \right] \frac{dV}{\vec{r}}$$

$$+ \frac{1}{4\pi} \int_{Vol} \left[2\, \overline{\frac{\partial \overline{u}_k}{\partial x_m} \frac{\partial u_m'}{\partial x_k}\Bigg|_{\vec{r}} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)_{\vec{x}}} \right] \frac{dV}{\vec{r}}$$

The standard interpretation of this is that there is a "rapid" and "slow" part

The first term contains only turbulence quantities and is termed "return to isotropy" or "slow distortion" T_{ij}^s

For $i=j$, $T_{ij}^s = 0 = T_{ij}$

Things we say in modeling 2nd order closure

Turbulence quantities are local functions of $\overline{u_i'u_j'}$, K, ε, $\overline{U_i}$, etc

Consistent in symmetry

Turbulent phenomena characterized by single scale based on K, ε

Small eddies isotropic

So say $T_{ij}^s = -\frac{c}{t} u_i'u_j'$ ← decay

but require $T_{ii}^s = 0$

$$\text{so } T_{ij}^{s} = -\frac{c}{t}\left(\overline{u_i'u_j'} - \frac{2}{3}\delta_{ij}k\right)$$

$$= -C\frac{\varepsilon}{K}\left(\overline{u_i'u_j'} - \frac{2}{3}\delta_{ij}k\right) \quad \text{(Pope eq. 11.24)}$$

This is earliest model (Rotta, 1951)

Advanced ideas: Real process of redistribution is nonlinear & modeling should reflect this

<u>Pope 11.3.3</u>

Not an easily modeled term (See Chen & Jaw "Fundamentals of Turbulence Modeling" Taylor & Francis 1998 pg 28-31

C has taken on wide range of values.

TOPS

"Rapid" Pressure strain "rapid distortion" "rapid return to isotropy"

$$T_{ij}^{r} \quad \frac{1}{4\pi}\int_{Vol}\left[2\,\frac{\partial \overline{U}_k}{\partial x_m}\,\overline{\frac{\partial u_m'}{\partial x_k}\Big|_{\vec{r}}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)\Big|_{\vec{x}}}\right]\frac{dV}{\vec{r}}$$

Named "rapid" since there is an instantaneous response in this correlation to mean velocity gradients

Simplest modeling approach

Approximate T_{ij}^{r} by shrinking down size of integration volume:

$$T_{ij}^{r} = C\,\frac{\partial \overline{U}_k}{\partial x_m}\,\overline{\frac{\partial u_m'}{\partial x_k}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}\,\frac{\ell^3}{\ell}$$

The assumption here is all turbulent transport quantities are local functions of k, ε, $\overline{u_i'u_j'}$, $\overline{U}$, etc

For proper symmetry, $T_{ii}^{r}=0$, and correct scaling we can write:

$$T_{ij}^{r} = C\left(\underbrace{\frac{\partial \overline{U}_j}{\partial x_m}\overline{u_m'u_i'}+\frac{\partial \overline{U}_i}{\partial x_m}\overline{u_m'u_j'}}_{-P_{ij}} - \frac{2}{3}\delta_{ij}\underbrace{\frac{\partial \overline{U}_k}{\partial x_m}\overline{u_m'u_k'}}_{P \text{ in K.e. equation}}\right)$$

$$T_{ij}^{r} = -C\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right)$$

For high strain

$$(2\overline{S}_{ij}\overline{S}_{ij})^{1/2} \gg \frac{\varepsilon}{k}, \quad T_{ij}^{r} \text{ dominates}$$

See RDT, (Pope, pg. 404

General modeling more sophisticated than above.

Boundary Conditions:

At walls:

No slip applies

Problems: 1) Steep gradients $\Rightarrow$ require very high resolution

2) Viscous effects important & high Re turbulence models (i.e. $\nu_t \gg \nu$) not applicable

A Solution: Use empirical laws to connect wall conditions (i.e. wall shear stress) to dependent variables outside viscous sublayer

From momentum equation, near wall velocity profile can be approximated by

$$u^+ = \frac{1}{K} \ln y^+ + C$$

$$u^+ = \frac{u}{u_\tau} \qquad u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2} \qquad \tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{wall}$$

$$y^+ = \frac{u_\tau y}{\nu}$$

Very near wall ($y^+ \lesssim 10$), viscous effects dominate

For $y^+ > 10$, can write $u^+ = \frac{1}{K} \ln E y^+$ (*)

$E \approx 9$ for smooth walls

Solve regular k-ε equations in turbulent zone

Use wall f^n for near wall & to get u at first mesh point

Solution B:

Wall models not alway adequate

e.g. separated flows
unsteady flows
transitional Re

Cannot predict things directly near wall

Low Reynolds # approach using wall damping

Typical set

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} k$$

where $\nu_t = C_\mu f_\mu \frac{k^2}{\bar{\varepsilon}}$ ← damping function ← Tabulated for various models in Chen & Jaw pg. 117-118

$\varepsilon = \bar{\varepsilon} + D$ ← function involving ν, k & gradients of k

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left(C_k f_\mu \frac{k^2}{\bar{\varepsilon}} \frac{\partial k}{\partial x_j} + \nu \frac{\partial k}{\partial x_j} \right) - \overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon$$

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left(C_\varepsilon f_\mu \frac{k^2}{\bar{\varepsilon}} \frac{\partial \bar{\varepsilon}}{\partial x_j} + \nu \frac{\partial \bar{\varepsilon}}{\partial x_j} \right) - C_{\varepsilon 1} f_1 \frac{\bar{\varepsilon}}{k} \overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2} f_2 \frac{\bar{\varepsilon}^2}{k} + E$$

$$\left. \begin{array}{l} f_\mu \\ f_1 \\ f_2 \end{array} \right\} \text{involve } y, y^+, \nu, k \ldots$$

Several variants exist