Solutions to Turbulent flows

Difficulties:

Coupled, nonlinear pdes → Large range of length & time scales
Complex boundary conditions

Approaches:

Analytical → not practical

Numerical

Full Direct Numerical Simulation
Only practical for low Re turbulence

"Reduced" Equations
Reynolds averaging
Large eddy simulation

Stochastic Solutions
Probability density function methods

→ Require "Modeling"

What do we mean by modeling?

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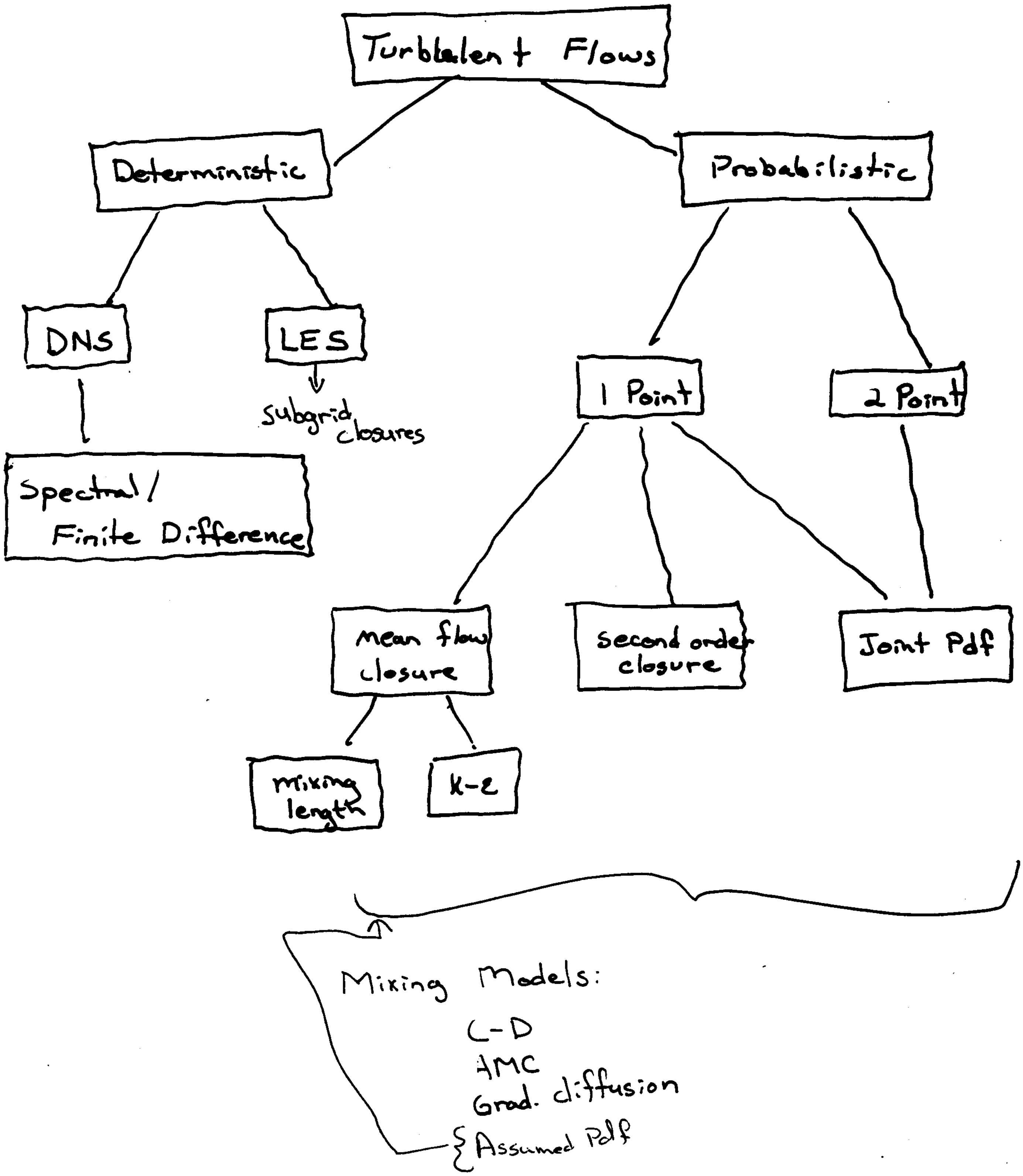
Classification of Simulation Approaches



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Turbulence Modeling:

Goal: Obtain a computationally treatable set of equations that accurately describe characteristics of turbulent flows

"Mean" momentum

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_i' u_j'}\right)$$

(For const. ρ, μ)

Knowledge of "mean" properties desirable

But what about $\overline{u_i' u_j'}$?

We have no prior knowledge, no equation to describe it. This is the

"Closure Problem"

Determination of $\overline{u_i' u_j'}$ (and other turbulence correlations) is a huge part of what "turbulence modeling" is all about.

Wont spend lots of time on specific models. Instead will present general ideas common in turbulence modeling

Fundamental idea in turbulence modeling of Reynolds stress:

Turbulence results in faster transport

$\Rightarrow$ Assume $\overline{u_i' u_j'} \sim$ mean velocity gradient

Call constant of proportionality the "turbulent" or "eddy" viscosity:

i.e.,
$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \qquad \star$$

Comments: Last term assures $\frac{1}{2} \overline{u_i' u_i} = k$

Looks like a gradient driven diffusion process with ν replaced by ν_t

So mean momentum becomes:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \bar{u}_i = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left[(\nu_t + \nu) \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \right] - \frac{2}{3} k \delta_{ij}$$

assuming $\nu_t >> \nu$

Can absorb k in p:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \bar{u}_i = -\frac{\partial (\bar{p} + \frac{2}{3} k)}{\partial x_i} + \frac{\partial}{\partial x_j} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

But what is ν_t ?

Concerns: Is physics of $\star$ correct?

If not, wow.

If so, how to determine ν_t ?

Modeling eddy viscosity:

Eddy - molecular viscosity analogy

From kinetic theory of gasses:

Mol. Viscosity results from momentum exchange due to molecular collisions

$$\nu \sim V_m L$$

L — mean free path between molecules

V_m — speed of molecules

Extend idea to turbulent motions: "eddies" collide and exchange momentum. So say

$$\nu_t \sim V L_m$$

L_m — "mixing length"

V — characteristic velocity of turbulence

First formulated by Prandtl for shear layer

only mean velocity gradient is $\frac{\partial \bar{u}_1}{\partial x_2}$

Prandtle suggests $V = L_m \left| \frac{\partial \bar{u}_1}{\partial x_2} \right|$

$$\Rightarrow \nu_t = L_m^2 \left| \frac{\partial \bar{u}_1}{\partial x_2} \right|$$

relates ν_t to mean velocity gradient & a mixing length, L_m

L_m needs to be specified

L_m from experimental data

Approach most useful thin shear layers

jets
wake
mixing layers } $L_m \sim \delta$ & constant across layer

Boundary layer – different values for L_m in different regions

Problems with Eddy Viscosity – Molecular Viscosity Analogy

Molecular Viscosity – elastic collisions (molecules retain shape)

mean free path large compared to molecular sizes

Turbulent eddies – don't retain shape

eddy size & interaction lengths same scale

So should treat $\nu_t \sim V L_m$ as definition (not physically derived law)

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Improvements to mixing length model

Goal - add more physics

Take turbulence property, $k = \frac{1}{2}\overline{u_i' u_i'}$ to define V

$$V = k^{1/2}$$

$$\text{\&} \quad \nu_t = C_\mu k^{1/2} L$$

How to determine k?

Solve equation for k. We derived earlier:

Can be expressed as (See sec 9 of notes - equations & text leading to 9.37)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i' u_i'}\right) + \overline{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i' u_i'}\right) = -\frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j' u_j'\right)}$$

$$- \overline{u_i' u_j'}\frac{\partial \overline{u}_j}{\partial x_i} \qquad (s_{ij}\, r_{ij})(s_{ij} + r_{ij})$$

$$+ \frac{1}{2}\nu \frac{\partial^2}{\partial x_i \partial x_i}\overline{k^2} \underbrace{- \nu \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}}}_{\text{often call this } \varepsilon}$$

results from manipulation

but $\varepsilon = 2\nu \overline{s_{ij} s_{ij}}$ difference not large

What has all this gained us?

Can't solve because we've introduced additional unknowns

$$\overline{u'_i\left(\frac{p'}{\rho} + \frac{1}{2}u'_j u'_j\right)}$$ ← turbulent diffusive transport

Model as $C_2 \nu_t \frac{\partial k}{\partial x_i}$ ← turbulent transport acts in gradient diffusive manner (hypothesis)

Model dissipation as

$$\varepsilon = C_3 \frac{k^{3/2}}{L}$$ this is k/t from $\varepsilon \sim U^3/L$

need time scale, $t = k^{1/2}/L$

Still Need to define L. Not straight forward

(Production term closed) $-\overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$ (once L specified)

because $u_i' u_j' \sim 2\nu_T \overline{S_{ij}}$

$$\nu_T = C_\mu k^{1/2} L$$

$$-\overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j} = 2 C_\mu k^{1/2} L \, \overline{S_{ij}} \frac{\partial \bar{u}_i}{\partial x_j}$$

Two-Equation Models

Need to define or determine length scale, L
Can determine from scale analysis. Need to use turbulence properties.

Already have k, Therefore can for length scale from

$$L \sim k^p Z^q$$ where Z is some property or quantity representative of turbulence

or some Z such that $\boxed{Z = k^n L^m}$

Several quantities have been proposed
frequency, ω
dissipation rate, ε

$k-\omega$ model
$k-\varepsilon$ model

$k-\varepsilon$ model: Probably most common 2-equation model.
Let $Z = \varepsilon \approx k^{3/2}/L$
Derivation of equation for ε $\left(2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}\right)$

Take $2\nu \frac{\partial u_i'}{\partial x_j} \times$ N.S.

↑ decompose into mean + fluctuating components

average the whole thing

This process gives

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = -2\nu \frac{1}{\rho} \frac{\partial}{\partial x_i}\left(\overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial p'}{\partial x_j}}\right) \quad (1)$$

$$- \nu \frac{\partial}{\partial x_k}\left(\overline{u_k' \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}\right) \quad (2)$$

$$- 2\nu \overline{u_k' \frac{\partial u_i'}{\partial x_j}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} \quad (3)$$

$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_k}} \frac{\partial \bar{u}_k}{\partial x_j} \quad (4)$$

$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \frac{\partial \bar{u}_i}{\partial x_k} \quad (5)$$

$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_k'}{\partial x_j}} \quad (6)$$

$$- 2\nu^2 \overline{\left(\frac{\partial^2 u_i'}{\partial x_j \partial x_k}\right)^2} \quad (7)$$

$$+ \mu \frac{\partial^2 \varepsilon}{\partial x_k \partial x_k} \quad (8)$$

(1) Turbulent transport – pressure fluctuations

(2) " " – turbulent fluctuations

(3) (4) (5) : Production of dissipation

(6) Dissipation of dissipation

(7) " " "

(8) Diffussion of dissipation

Lots of unclosed terms, how do we treat?

Lots of terms to model

Most people treat equation as empirical & don't attempt to model detailed behavior of each term. We start with the following:

$$\frac{D\bar{\varepsilon}}{Dt} = \text{Production} + \text{Turbulent Transport} + \text{dissipation}$$

Production: $\sim \underbrace{\text{K.e. production}} * t^{-1}$

$$\overline{u_i' u_j'} \frac{\partial \bar{u}_j}{\partial x_i} \frac{\varepsilon}{K} C$$

$$\nu_t 2\bar{S}_{ij} \frac{\partial \bar{u}_j}{\partial x_i} \frac{\varepsilon}{K} C$$

$$\nu_t = C_\mu \frac{K^2}{\varepsilon} \Rightarrow$$

$$\text{Production} = C_\mu \frac{K^2}{\varepsilon} 2\bar{S}_{ij} \frac{\partial \bar{u}_j}{\partial x_i} \frac{\varepsilon}{K} C$$

$$= C_{1\varepsilon}\, 2K\, S_{ij} \frac{\partial \bar{u}_j}{\partial x_i}$$

Turbulent transport:

Combine pressure & velocity fluctuations

$$2\nu \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial p'}{\partial x_j}\right) + \nu \frac{\partial}{\partial x_k}\left(u_k' \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right)$$

$$\approx \frac{\partial}{\partial x_i}\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i} \quad \leftarrow \text{gradient transport}$$

Dissipation of dissipation

$$\sim \varepsilon/t \sim \varepsilon \frac{\varepsilon}{k} = C_{2\varepsilon}\frac{\varepsilon^2}{k}$$

See web notes for constants

How to determine?

Decaying grid turbulence (homogeneous)

$$\left.\begin{aligned} \frac{\partial \varepsilon}{\partial t} &= -C_{2\varepsilon}\frac{\varepsilon^2}{k} \\ \frac{\partial k}{\partial t} &= -\varepsilon \end{aligned}\right\} \text{used to determine } C_{2\varepsilon}$$

Homogeneous steady turbulence

$$2\nu_t \overline{S_{ij}}\frac{\partial \overline{u_i}}{\partial x_j} = \varepsilon \qquad \text{k.e. production} = \text{dissipation}$$

$$C_{1\varepsilon} 2k S_{ij}\frac{\partial \overline{u_i}}{\partial x_j} = C_{2\varepsilon}\frac{\varepsilon^2}{k}$$

Comment on energy budgets

See __________ for details on experiments to determine constants

K-ω model

As discussed earlier any property of the form
$Z = k^n L^m$ can be used to derive a
length scale. Of coarse we require some physical significance of Z that relates to turbulence.

Probably the earliest 2 equation model was a k-ω model, described by Kolmogorov

Basic postulated idea

ν_T is porportional to k

$\frac{\nu_T}{k}$ has dimensions of t

so a quantity with dimensions of time can be used to determine ν_T ; $\nu_T \sim k/t = k\omega$

$\omega = c\, k^{1/2}/\ell$

$\frac{1}{\omega}$ time scale on which dissipation of k.e. occurs
→ set by properties of large scales

Kolmogorov defined ω as dissipation per unit kinetic energy, ε/k

The transport equation for ω has undergone a 60 year history of development

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First put forth by Kolmogorov 1942
Saffman 1970
Wilcox et al 70's 80's, 90's
Speziale et al 90
Peng et al 90's

Write equation in terms of basic physical processes governing motion in fluid:

convection
dissipation
production
turbulent transport (dispersion)
diffusion

$$\frac{\partial \omega}{\partial t} + \overline{u_j}\frac{\partial \omega}{\partial x_j} = C_a \frac{\omega}{K} \overline{u_i' u_j'} \frac{\partial \overline{u_i}}{\partial x_j'} \quad \text{production}$$

$$- C_b \omega^2 \quad \text{dissipation}$$

$$+ \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \omega}{\partial x_j}\right) \quad \text{molecular diffusion}$$

$$+ \frac{\partial}{\partial x_j}\left(C_3 \nu_t \frac{\partial \omega}{\partial x_j}\right) \quad \text{turbulent transport}$$

Also, in kinetic energy equation, dont have ε (or must model)

$$\varepsilon = C_4 K\omega$$

& must determine coefficients.

In Wilcox 1998 k-ω model, these are determined

k-ε, k-ω differences:

Boundary conditions - can arbitrarily specify ω on solid surfaces

k-ω appears to give better results for wall bounded flows.

Particularly for rough wall flows

Some General Comments on this:

Basic idea: Provide description of diffusive nature of turbulence in terms of eddy viscosity

Assume eddy viscosity $\sim$ LV

Use turbulence quantities to determine L, V

($k-\varepsilon$, $k-\omega$)

Derive or devise transport equations for quantities k, ε (assume behavior)

Model unclosed terms

Determine coefficents by comparison with experiments

Are physics of $\nu_t \bar{S}_{ij}$ correct?

When?

How about approximations for estimating ν_t?

Can we get around these assumptions?

Recall exact equation:

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(-\overline{u_i' u_j'} + \frac{\partial}{\partial x_j}\bar{u}_i\right)$$

Why not derive transport equation for $\overline{u_i' u_j'}$