Length & Time Scales

Smallest length scale of the velocity field

Kolmogorov Scale $\cong \eta$

Small scales depend only on $\nu\left(\frac{m^2}{sec}\right)$ & $\varepsilon\left(\frac{m^2}{sec^3}\right)$

Form a length scale from these quantities

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \qquad \bigstar$$

Relate to large scales:

For steady flow, energy fed in at large scales = energy dissipated at small scales

$$\varepsilon \sim \frac{U^2}{t} = \frac{U^2}{L/U} \sim \frac{U^3}{L} \qquad \bigstar\bigstar$$

Use $\bigstar\bigstar$ in $\bigstar$ gives

$$\eta = \left(\frac{\nu^3 L}{U^3}\right)^{1/4}$$

Then $\frac{L}{\eta} \approx \left(\frac{UL}{\nu}\right)^{3/4} = Re^{3/4}$

This uses kolmogorov hypothesis to get estimate for η

"Kolmogorov scalings"

Taylor Microscale: λ (See pope pg 198)

Used to characterize fluctuating strain field

$$\left(\frac{\partial u'}{\partial x}\right) = \frac{u'^2}{\lambda} \quad \star$$

Related to dissipation in isotropic turbulence

$$\varepsilon = 15 \nu u'^2 / \lambda_g^2$$

Let $L \approx k^{3/2}/\varepsilon$ (characterization of Large eddies

then $\lambda/L = \sqrt{10}\, Re_L^{-1/2}$

Doesn't have well defined physical interpretation

Other Definitions

Common to define length scales in terms of auto correlation function

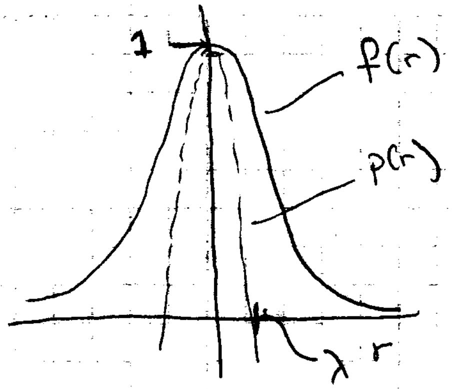
$$f(r,t) \equiv u_1(\vec{x})\, u_1(\vec{x} + r x_1) \,/\, \overline{u_1^2}$$

$$g(r,t) = u_2(\vec{x})\, u_2(\vec{x} + r x_1) \,/\, \overline{u_2^2}$$

$L_{11} \equiv$ longitudinal integral scale $= \int_0^\infty f(r,t)\, dr$

Taylor microscale $\lambda \equiv \left[-\frac{1}{2}\frac{d^2 f}{dr^2}(r=0)\right]^{-1/2}$

Huh?



p(r) is parabola with

$$\left.\begin{aligned} p(0) &= f(0) \\ p'(0) &= f'(0) \\ p''(0) &= f''(0) \end{aligned}\right\} \Rightarrow p(r) = 1 + \tfrac{1}{2} f''(0)\, r^2 = 1 - r^2/\lambda^2$$

See pope for relation to $\star$

Time Scales:

Large eddies: $t_L = L/U$ (Large eddy "turnover" time)

small scales, from ν, ε form time scale

$$t_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

Using $\varepsilon \approx U^3/L$ gives

$$t_L / t_\eta \sim Re^{1/2}$$

Length & time scales of scalar field: will discuss later.

Time for eddy to be reduced to Kolmogorov Scale
(Time for energy at integral scales to be dissipated at small scales). Call this τ_L

Relationship between t_ℓ (time scale of size ℓ eddy) $\equiv \ell/u$ to τ_ℓ

In general, for size ℓ eddy

$$\frac{d\ell}{dt} \sim \frac{\ell}{t_\ell} \quad \star$$

at large scales k.e. flux $\approx U^2/L/U = \underbrace{\frac{U^3}{L} \approx \frac{u^3}{\ell} \approx \varepsilon}_{\text{independent of } k}$ $\star\star$

rewrite $\star\star$ as

$$\frac{U^3}{L}\frac{L^2}{L^2} \sim \frac{u^3}{\ell}\frac{\ell^2}{\ell^2} \quad \text{or} \quad \frac{t_\ell^3}{\ell^2} \sim \frac{t_L^3}{L^2}$$

$$\text{or} \quad t_\ell \sim \left(\frac{\ell}{L}\right)^{2/3} t_L \quad \star\star\star$$

Using $\star\star\star$ in $\star$ gives

$$\frac{d\ell}{dt} \sim \frac{\ell^{1/3} L^{2/3}}{t_L}$$ separate variables:

$$L^{-2/3} \int_L^\eta \ell^{-1/3}\, d\ell = \int_0^{\tau_L} \frac{dt}{t_L}$$ τ_L time to go from $L \to \eta$

Solving gives: $$\frac{\tau_L}{t_L} = 1 - \left(\frac{\eta}{L}\right)^{2/3} = 1 - Re^{-1/2}$$

Statistical Representation of the flow
(See notes, Turbulent Transport)

Flow decomposition

$$u_i = \overline{u} + u'$$

$$p = \overline{p} + p' \qquad \star$$

$$\sigma_{ij} = \overline{\sigma_{ij}} + \sigma_{ij}'$$

u_i	$=$	$\overline{u}$	$+$	u'
Full time dependent fluctuating velocity field		~~mean~~ "average" velocity		fluctuation about mean

$\overline{u}$ can be $f(x)$ or $f(t)$, depending on how ~~mean~~ "average" is defined and conditions of flow

$$u' = f(x,t)$$

$\star$ is a definition and conceptualization

$$\overline{\sigma_{ij}} = -p\delta_{ij} + \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

$$\sigma_{ij}' = -p'\delta_{ij} + \mu\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

Use definitions in ✱ in N.S. & average equations

$$\rho \frac{\partial \overline{\bar{u}_i + u_i'}}{\partial t} + \rho \frac{\partial \overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\bar{\sigma}_{ij} + \sigma_{ij}'}\right)$$

With $\overline{u'} = 0$, $\overline{\bar{u}u'} = 0$ $\bar{\bar{u}} = \bar{u}$ (for all Variables, we have:

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial (\bar{u}_i \bar{u}_j + \overline{u_i' u_j'})}{\partial x_j} = \frac{\partial}{\partial x_j} \overline{\sigma_{ij}}$$

Equation for $\dot{\overline{u_i}}$

Similar to N.S., except $\overline{u_i' u_j'}$

What is this?

All effects of turbulence on mean flow contained in this term. (In this "man-made" mathmatical representation

Momentum flux associated with stress in fluid:

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\sigma_{ij}} - \overline{u_i' u_j'}\right)$$

So $\overline{u_i' u_j'}$ termed "Reynolds Stress Tensor"

$$= \frac{\partial}{\partial x_j}\left(-\bar{p}\,\delta_{ij} + 2\nu \bar{S}_{ij} - \overline{u_i' u_j'}\right)$$

Equation is still exact, but we don't have information on $\overline{u_i' u_j'}$.

Scale analysis for Estimate of Reynolds Stress

$$\overline{S}_{ij} \sim \nu \frac{U_t}{L}$$

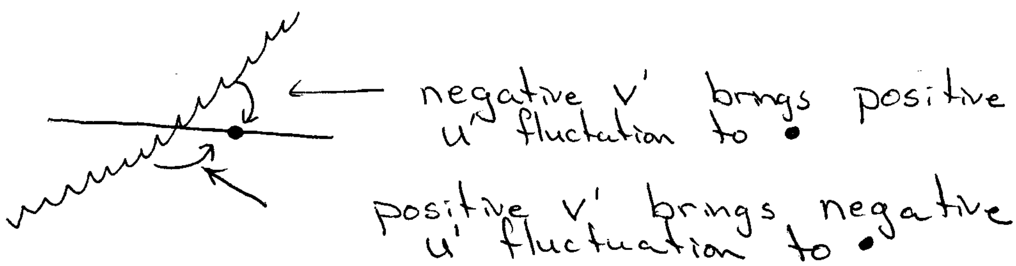
$$\overline{u_i' u_j'} \sim U_t^2$$

$$\text{Ratio:} \quad \frac{\overline{u_i' u_j'}}{\overline{S}_{ij}} \sim \frac{U_t L}{\nu} = Re_{U_t}$$

For high Re flows $\overline{u_i' u_j'} \gg \overline{S}_{ij}$

In a turbulent flow with mean gradient,

$\overline{u_i' u_j'}$ usually negative



Whats the real mechanisms here?

Turbulent K.E.

Full K.E. (incomp., const. coef.)

$$\frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) + \frac{\partial}{\partial x_j}\left(u_j \frac{u_i u_i}{2}\right) = \underbrace{-\frac{1}{\rho}\frac{\partial u_j p}{\partial x_j}}_{①} + \underbrace{\frac{\partial}{\partial x_j} u_i 2\nu S_{ij}}_{②} \underbrace{- 2\nu S_{ij}\frac{\partial u_i}{\partial x_j}}_{③}$$

From momentum, multiply by u_i, manipulate.

① Work done by pressure forces

② " " " viscous stresses

③ dissipation of K.E

(See ME 7700 or other advanced fluid notes)

Can also write as:

$$\frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) = -\frac{\partial}{\partial x_j} u_j \underbrace{\left(\frac{p}{\rho} + \frac{u_i u_i}{2}\right)}_{①} + \frac{\partial}{\partial x_j} u_i 2\nu S_{ij} - 2\nu S_{ij}\frac{\partial u_i}{\partial x_j} \quad \star$$

① now interpretated as work due to total ~~dynamic~~ pressure

Next, decompose into mean + fluctuation:

$$u_i u_i = (\bar{u}_i + u_i')(\bar{u}_i + u_i') = \bar{u}_i \bar{u}_i + 2\bar{u}_i u_i' + u_i' u_i' \quad \star\star$$

Using this ($\star\star$) and other decompositions ($\star$ pg 1) in $\star$

gives;

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{u}_i \bar{u}_i\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i' u_i'}\right)$$

$$= -\frac{\partial}{\partial x_j}\bar{u}_i\left(\frac{\bar{P}}{\rho} + \frac{1}{2}\bar{u}_j \bar{u}_j\right) + \nu\frac{\partial}{\partial x_i}\bar{u}_j 2\bar{S}_{ij} - \nu 2\bar{S}_{ij}\frac{\partial \bar{u}_j}{\partial x_i}$$

$$-\frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j' u_j'\right)} - \frac{\partial}{\partial x_i}\bar{u}_j \overline{u_i' u_j'} - \frac{1}{2}\frac{\partial}{\partial x_i}\bar{u}_i \overline{u_j' u_j'}$$

$$+ \nu\frac{\partial}{\partial x_i}\overline{u_j' 2 s_{ij}} - \overline{\nu 2 s_{ij}\frac{\partial u_j'}{\partial x_i}} \quad \star\star\star$$

Turbulent K.E. (cont.)

Mean K.E. Equation:

Start with mean momentum, multiply each term by $\overline{u}_i$, and rearrange to give:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u}_i\overline{u}_i\right) + \frac{\partial}{\partial x_j}\overline{u}_j\left(\frac{\overline{p}}{\rho} + \frac{1}{2}\overline{u}_i\overline{u}_i\right)$$

$$= \overset{(1)}{-\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j}} + \overset{(2)}{\frac{\partial}{\partial x_j}\left(-\overline{u_i'u_j'}\,\overline{u}_i\right)}$$

$$+ \overset{(3)}{\nu\frac{\partial}{\partial x_i}\overline{u}_j\, 2\overline{S}_{ij}} - \overset{(4)}{\nu\, 2\overline{S}_{ij}\frac{\partial \overline{u}_i}{\partial x_j}}$$

****

(1) deformation work

(2) work by turbulent stresses

(3) work by viscous stresses

(4) dissipation

Finally, turbulent k.e. equation obtained from subtracting **** from ***:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'u_i'}\right) + \overline{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'}\right) = \underset{(1)}{-\frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j'u_j'\right)}}$$

$$\underset{(2)}{-\overline{u_i'u_j'}\frac{\partial \overline{u}_j}{\partial x_i}} + \underset{(3)}{\nu\frac{\partial}{\partial x_i}\overline{u_i'\, 2 s_{ij}}} - \underset{(4)}{\nu\,\overline{2 s_{ij}\frac{\partial u_i'}{\partial x_j}}}$$

(1) Turbulent transport of ~~me~~ mechanical energy

(2) Deformation work on mean flow

(3) viscous work by turbulent shear stresses

(4) dissipation

Note that $s_{ij} \frac{\partial u_i'}{\partial x_j} = s_{ij} s_{ij}$

& $\overline{S}_{ij} \frac{\partial \overline{u}_i}{\partial x_j} = \overline{S}_{ij} \overline{S}_{ij}$ (Right ?)

Assuming a statistically steady flow, equations for mean & kinetic energy are:

<u>Mean</u>

$$\overline{u}_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\overline{u}_i\overline{u}_i\right) = \frac{\partial}{\partial x_j}\left(-\frac{\overline{P}}{\rho}\overline{u}_j + 2\nu\,\overline{u}_i\overline{S}_{ij} - \overline{u_i'u_j'}\,\overline{u}_i\right)$$

$$+ 2\nu\,\overline{S}_{ij}\overline{S}_{ij} + \overline{u_j'u_i'}\,\overline{S}_{ij}$$

<u>TKE</u>

&

$$\overline{u}_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\overline{u_i'u_i'}\right) = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j'p'} + \frac{1}{2}\overline{u_i'u_i'u_j'} - 2\nu\,\overline{u_i's_{ij}}\right)$$

$$- \overline{u_i'u_j'}\,\overline{S}_{ij} - 2\nu\,\overline{s_{ij}s_{ij}}$$

Note $\overline{u_j'u_i'}\,\overline{S}_{ij}$ → opposite sign in each equation

This is deformation work, called "production"

For negative $\overline{u_i'u_j'}$, term takes from mean → to K.E.

For homogeneous turbulent flows, statistical properties don't change in space. In this case, tke equation becomes

$$\overline{u_i'u_j'}\,\overline{S}_{ij} = 2\nu\,\overline{s_{ij}s_{ij}}$$

production = dissipation

Passive Scalars

"Exact" transport (constant molecular diffusivity)

$$\frac{\partial \phi}{\partial t} + \frac{\partial u_j \phi}{\partial x_j} = D \frac{\partial^2 \phi}{\partial x^2}$$

$$\phi = \overline{\phi} + \phi'$$

$$u_i = \overline{u}_i + u_i'$$

Plug into equation & average gives

$$\frac{\partial \overline{\phi}}{\partial t} + \frac{\partial \overline{u}_j \overline{\phi}}{\partial x_j} = -\frac{\partial \overline{\phi' u_j'}}{\partial x_j} + D \frac{\partial^2 \overline{\phi}}{\partial x_j \partial x_j}$$

$$= \frac{\partial}{\partial x_j}\left(D \frac{\partial \overline{\phi}}{\partial x_j} - \overline{\phi' u_j'} \right)$$

Scalar variance

$$\frac{\partial \overline{\phi'^2}}{\partial t} + \overline{u}_j \frac{\partial \overline{\phi'^2}}{\partial x_j} = \overset{(a)}{\left[2\overline{u_j'\phi'}\, \frac{\partial \overline{\phi}}{\partial x_j} \right]} - \overset{(b)}{\frac{\partial}{\partial x_j} \overline{u_j' \phi'^2}}$$

$$+ \overset{(c)}{\frac{\partial}{\partial x_j} D \frac{\partial}{\partial x_j} \overline{\phi'^2}} - 2D \overset{(d)}{\overline{\frac{\partial \phi'}{\partial x_j} \frac{\partial \phi'}{\partial x_j}}}$$

(a) Production

(b) turbulent transport

(c) diffusion

(d) scalar dissipation

TOPS 35500

Direct Numerical Simulation - Talk about what it is

Large length / timescales

Computationally unfeasible until recently
Still limited to low Re flows

Used now to study ~~i~~ physical behavior

Not real engineering applications