Vorticity Dynamics

Turbulent flows all exhibit high levels of fluctuating vorticity. Much of the dynamics and structure of turbulent flows can be productively interpreted and analyzed in terms of the vorticity dynamics.

We will proceed as follows:

1) Define vorticity & interpret turbulent flows in terms of the vorticity
2) Derive the vorticity equation and discuss the various mechanisms of vorticity transpor:
3) Look at some examples that illustrate how we can use vorticity to interpret what's going on
4) Discuss details of vorticity transport
5) Introduce classical Equilibrium Range Theories

Vorticity: Curl of velocity field

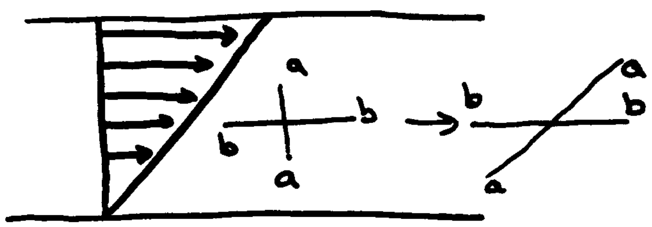
$$\vec{\omega} = \nabla \times \vec{u} \quad \text{or} \quad \omega_i = e_{ijk} \frac{\partial u_k}{\partial x_j}$$

$$\text{eg. } \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Q: what are other components?

Vorticity is defined as twice the local average rotation rate of two initially $\perp$ lines following a fluid element

Eg: Shear Flow (Couette)



Rotation of $a-a = -\frac{\partial u}{\partial y}$

{ See any undergrad fluids text for thorough discussion of this stuff }

Turbulence is rotational & characterized by high fluctuations in vorticity

Vorticity Equation

A transport equation for vorticity can be obtained by taking the curl of the momentum equation

The Navier Stokes Equations can be writen in the following form:

$\frac{\partial}{\partial x_j} u_i u_j$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right) - \underbrace{\varepsilon_{ijk}\, u_j \Big(\underbrace{\varepsilon_{klm} \frac{\partial u_m}{\partial x_l}}_{\nabla\times\vec{u}=\vec{\omega}}\Big)}_{\vec{u}\times\vec{\omega}} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

or in vector notation:

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\vec{u}\cdot\vec{u}\right) - \vec{u}\times(\nabla\times\vec{u}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\tau$$

Take curl ($\nabla\times$) of above equation:

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla\times\left(\nabla\tfrac{1}{2}\vec{u}\cdot\vec{u}\right) - \nabla\times(\vec{u}\times\vec{\omega}) = -\nabla\times\left(\frac{1}{\rho}\nabla p\right) + \nu\nabla^2\vec{\omega}$$

Now $\nabla\times\left(\nabla\tfrac{1}{2}\vec{u}\cdot\vec{u}\right) = 0$ (This is a vector identity, $\nabla\times\nabla\phi \equiv 0$ ← show this - use Einstein index notation)

Also:

$$\nabla\times(\vec{u}\times\vec{\omega}) = (\vec{u}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{u} + \vec{\omega}(\nabla\cdot\vec{u}) - \vec{u}\,\nabla\cdot\vec{\omega}$$

<u>Prove This</u>

Let's look at each of these terms

a) $(\vec{u}\cdot\nabla)\vec{\omega}$ or $u_j \frac{\partial}{\partial x_j}\omega_i$ <u>Convection</u>

This is just a convection term. The vorticity gets convected (moved around) by the flow field. In the full vorticity equation for one vorticity component, you would have:

$$u\frac{\partial \omega_1}{\partial x} + v\frac{\partial \omega_1}{\partial y} + w\frac{\partial \omega_1}{\partial z}$$

b) $(\vec{\omega}\cdot\nabla)\vec{u}$ or $\omega_j \frac{\partial u_i}{\partial x_j}$ <u>Vortex Stretching</u>

This can be argued to be the most important turbulence term. It has no counter part in any of the transport equations you have encountered yet. It represents enhancement of vorticity by stretching and is the mechanism by which turbulent energy is transfered to smaller scales. We will give this term a lot of attention.
(it's $=0$ in 2-D flow)

c) $\vec{\omega}(\nabla\cdot\vec{u})$ or $\omega_i \frac{\partial u_j}{\partial x_j}$ <u>Expansion</u>

~~Rep~~ Describes effects of expansion (or compression) on vorticity field

Compress fluid → increase local vorticity
Expand " → decrease " "

(<u>By redistribution of vorticity</u>!)

31 (Not necessarily making new vorticity)

d) $\vec{u}(\nabla\cdot\vec{\omega})$ or $u_i \frac{\partial \omega_j}{\partial x_j}$, $\{ \vec{u}\left(\nabla\cdot(\nabla\times\vec{u})\right)$

This is always $=0$ show it

Now consider $-\nabla\times\left(\frac{1}{\rho}\nabla p\right)$

This can be expanded out as

$$\frac{1}{\rho^2}\left(\nabla\rho\times\nabla p\right)$$

This term is called the baroclinic torque and results when density gradients and pressure gradients are not aligned.

In a uniform, constant density flow it is of coarse $=0$.

In reacting flows (with heat release) and flows with different compositions, this term can have an important impact on the flow field development

Finally $\nu\nabla^2\omega$ is just viscous diffusion of vorticity

Finally, we can write

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t} + \vec{u}\cdot(\nabla \vec{\omega})}_{\frac{D\vec{\omega}}{Dt}} - (\vec{\omega}\cdot\nabla)\vec{u} + \vec{\omega}(\nabla\cdot\vec{u})$$

$$= \frac{1}{\rho^2}(\nabla\rho \times \nabla p) + \nu\nabla^2\vec{\omega}$$

or $$\frac{\partial \omega_i}{\partial t} + u_j\frac{\partial \omega_i}{\partial x_j} - \omega_j\frac{\partial u_i}{\partial x_j} + \omega_i\frac{\partial u_j}{\partial x_j}$$

$$= \frac{1}{\rho^2}\varepsilon_{ijk}\frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k} + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

For constant density:

$$\frac{\partial \omega_i}{\partial t} + u_j\frac{\partial \omega_i}{\partial x_j} - \omega_j\frac{\partial u_i}{\partial x_j} = \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

For 2-D constant density

$$\frac{\partial \omega_i}{\partial t} + u_j\frac{\partial \omega_i}{\partial x_j} = \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \leftarrow$$ In this case vorticity just satisfies a standard convection-diffusion equation.

For high Re, 2-D flows, vorticity is a good indicator of fluid flow patterns.

Vortex Stretching $\left(\omega_i \frac{\partial u_i}{\partial x_j}\right)$

Essential to 3-D structure, development, & amplification of turbulence.

Vortex stretching can:

- Homogenize the flow
- Reduce length scale of fluctuations
- Mechanism of energy transfer to smaller scales

Let's start with a qualitative description:

Consider a fluid element with vorticity $\vec{\omega}$

① and ② are subject to random fluid perturbations
At some time t later we have

Separation between 1–2 increases
(random walk analogy)

$\Rightarrow$ element "stretches" & r decreases

<u>Angular momentum</u> $\sim \omega r^2$

$$r \searrow \Rightarrow \omega \nearrow \quad (\text{cons. of ang. mom.})$$

Stretching results in length scale reduction in direction $\perp$ to ~~flow~~ stretching while intensifying magnitude of vorticity.

<u>Kinetic Energy</u> $\sim \omega^2 r^2$

If ωr^2 is conserved, stretching results in an increase in K.e. (At the expense of K.e. that does the stretching)

"Big whorls have little whorls that feed on their velocity..."

As smaller scales are approached, viscous diffusion takes over

"... and so on to viscosity."

{ We need an external energy source or the turbulence will decay.

Another interpretation

$$\omega_j \frac{\partial u_i}{\partial x_j}$$

↑ Expand deformation tensor:

$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j S_{ij} + \omega_j R_{ij}$$

But $\omega_j R_{ij} \equiv 0$ (Show this)

So $\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j S_{ij}$ Interpretation: Amplification of Vorticity by local strain rate.

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

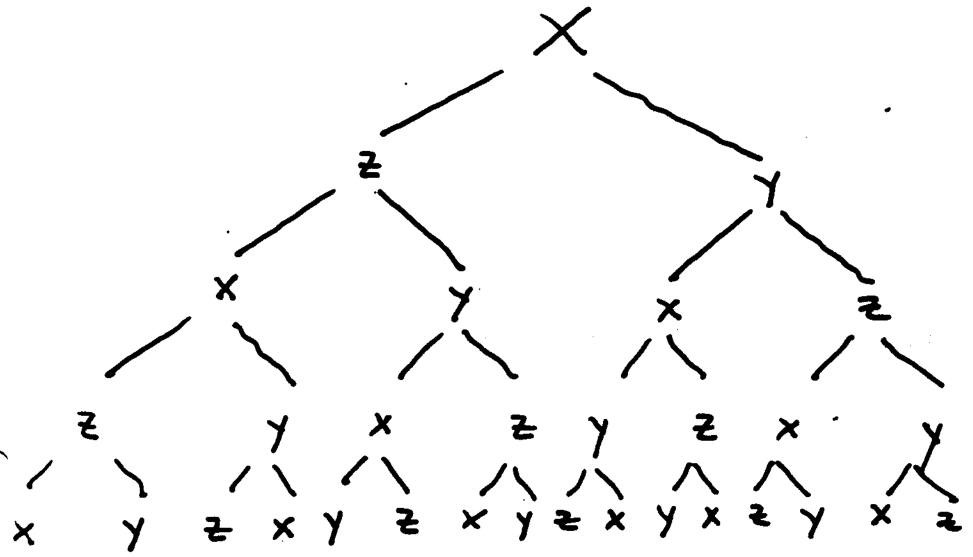
$$R_{ij} = \left(\quad - \quad \right)$$

$$\frac{\partial \omega}{\partial t} = \left(\text{vortex stretch}\right)$$

From Bradshaw:

Consider large scale motion in x

a) This intensifies motions in y & z at a smaller scale



b) Intensified z → stretching ⇒ intensifies x–y at small scale

" Y → " → " z–x " " "

⋮

As process continues, we have intense fluctuations in all directions (at the smaller scales)

⇒ directional preferences at large scales are obscured at small scales

{ Small scales tend to be isotropic (with important exception)

Back to Some Vorticity Dynamics Stuff
"Equilibrium Range Theories"

Kolmogorov's "Theory of Locally Isotropic Turbulence"

Length Scale Regimes:

L large (Integral) - energy maintained by mean flow

η - dissipation scale

$L \gg l_i \gg \eta$ ← since $l_i \gg \eta$, viscosity not yet important. Since $l_i \ll L$ orientation by mean flow destroyed by vortex breakdown mechanism.

<u>Kolmogorov's 1st Similarity Hypothesis:</u>

At sufficiently high Reynolds # there is a range of wave numbers (high) where the turbulence is statistically in equilibrium and uniquely determined by the parameters ε and ν

Deduced from the idea that for sufficiently small scale fluctuations, a homogeneous, isotropic, steady statistical regime will exist. This regime is characterized by the presence of a mean energy flux, and the dissipation of this flux by viscosity.

$\Rightarrow$ for small scale fluctuations, turbulence will be determined by ε and ν

(Also reasoning behind derivation of $\eta = (\nu^3/\varepsilon)^{1/4}$

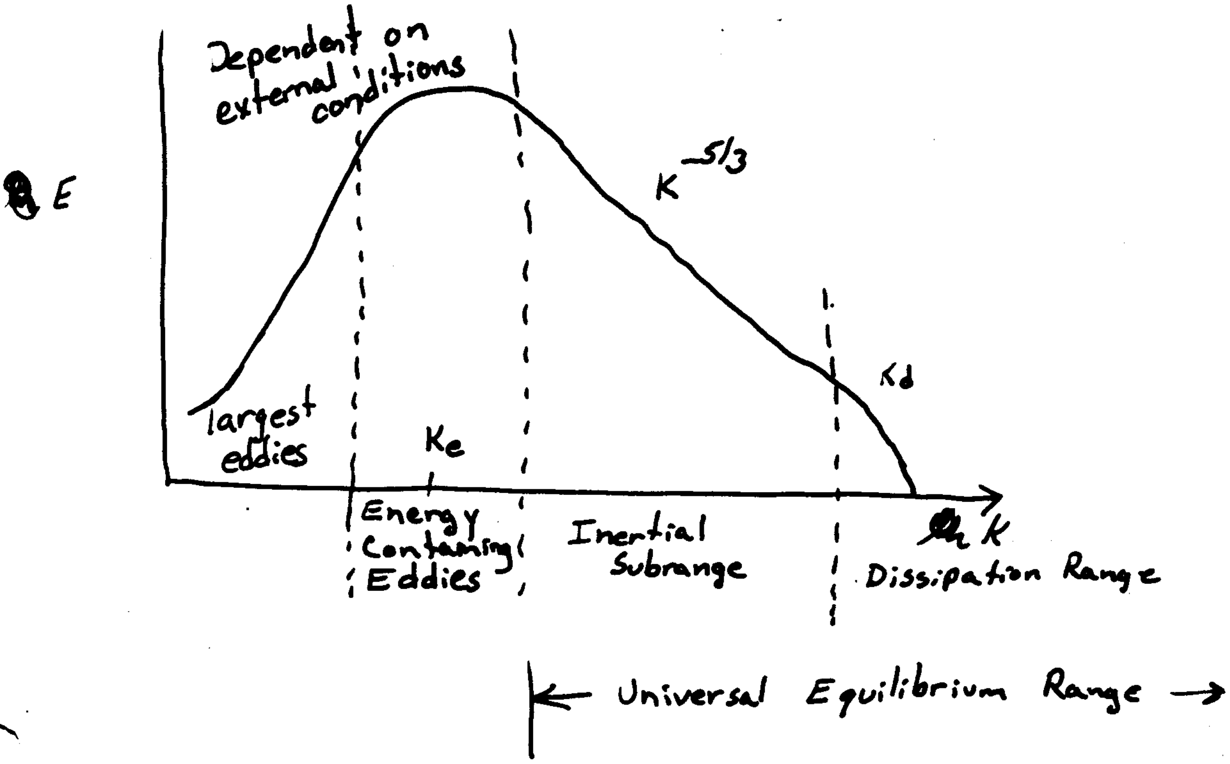
Now in l_i ($L \gg l_i \gg \eta$) viscosity won't be important since $l_i \gg \eta$, yet flow is still isotropic ($l_i \ll L$). This leads to

Kolmogorov's 2nd Similarity Hypothesis:

If the Reynolds number is sufficiently large, there exists a range of wavenumbers, K_i ($K_L \ll K_i \ll K_\eta$) where the turbulence is independent of ν and is unambiguosly defined by the value of the dissipation, ε

In this range, the inertial transfer of energy is the primary parameter characterizing the turbulence. This is called the "inertial subrange."

..neric Energy Spectrum:



Form of Energy Spectrum in Inertial subrange:

Only depends on ε, Dimensionally, we must have

$$\boxed{E(k) = C\,\varepsilon^{2/3} k^{-5/3}}$$

C = Kolmogorov Constant

In other regimes must solve dynamic equation for E

i.e. $\frac{\partial}{\partial t} E(k,t) = F(k,t) - 2\nu k^2 E(k,t)$

There is a k^4 region as $k \to 0$