ME 7960

Turbulence

Characteristics of a Turbulence Course

Methods of Description
math tools

Physics of Turbulent Flows
Describe, characterize, understand
physical mechanisms

→ Specific focus:
Mixing
Momentum Transport
Flow Geometries
Channels
Boundary Layers
Free Shear Layers

Modeling of Turbulence
Predict behavior

→ Methods
RANS
LES
PDF
Analytical
LEM
ODT

Turbulence Plays an Essential role in CFD

$\boxed{\text{CFD}}$

→ Mesh Generation
→ Algorithm Development
→ Turbulence Modeling

Turbulence dominates flows of Many Engineering Application:

External Flows around cars
planes
etc.

Flow in Pipes

Mixing in tanks, pipes

Geophysics: Cloud motion Planetary B.L.
Rivers
Ocean

Definitions of Turbulence: Lots of them

Characteristics of Turbulent Flows:

- Described by N-S.
- Enhanced diffusivity
- Dissipative
- 3-D
- Large range of time and length scales
- Random ...
- ... Order
- Unpredictable (Impossible to predict instantaneous flow)
- Unsteady

others?

Laminar vs. Turbulent Flow

Laminar - smooth, ordered

Turbulent - see above

Distinction usually obvious (but not always)

Flow characteristics change dramatically when flow becomes turbulent

Mixing is Fast

Lift & Drag Affected

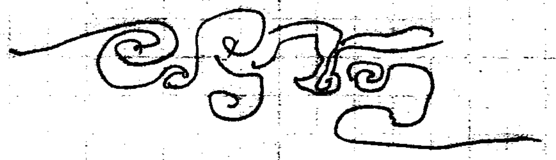
Laminar Flow - Viscosity smooths out perturbations

Turbulent Flow - Additional Forces (Inertia, Buoyancy) upset equilibrium ⇒ complex, unsteady flows.

Inertia $\sim VL^2$

Viscous $\sim \nu L$

Highly viscous – Laminar 

High inertia – Turbulent 

High inertia $\Rightarrow$ fluid cannot respond rapidly enough to viscosity $\Rightarrow$ breaks up into eddies.

$$\frac{\text{inertia}}{\text{viscous}} = Re = \text{Reynolds \#} = \frac{VL}{\nu}$$

Increase inertial Forces
$\rightarrow$ give smaller eddies
$\rightarrow$ until viscous forces take over

Gives finite minimum eddy size

Ideas: 1) Energy cascade

K.E. migrates through wave # spectrum.
Then dissipated by friction

2) Wide range of length and time scales.

Needs a source of energy

Length & time scales in Turbulent Flows

Turbulent Flows exhibit a wide range of length & time scales. This has a lot of implications regarding our ability to predict the flow

Largest eddy (length scale) = L (Integral Scale)
Geometric Constraint

Smallest eddy = η (Kolmogorov Scale)
viscosity

Q: What is $\frac{L}{\eta}$? See notes section 6 for analysis

$$\frac{L}{\eta} \sim Re^{3/4}$$

(chemical length scales can be $\ll \eta$)

Time scales:

Different fluid motions are associated with different time scale

$\tau_L \sim$ "large"

$\tau_\eta \sim$ "small"

$$\frac{\tau_L}{\tau_\eta} \sim Re^{1/2}$$

Some Implications of all this (for future reference)

A) On Numerical Simulation of Turbulent Flow:

For most flows, Re is Big so $Re^{3/4} = \text{Big}^{3/4} = \text{big}$

$\Rightarrow \frac{L}{\eta} = \text{big}$!

All length scales are dynamically important so in a <u>full</u> numerical simulation we must resolve all length scales

But turbulence is 3-D & unsteady $\left(\frac{t_L}{t_\eta} \sim Re^{1/2} \right)$

Take $Re = 10^5$ (that's reasonable)

Assume an eddy at Kolmogorov scale can be described by 1 grid point in a finite difference discretization

Then we need $\left[(10^5)^{3/4} \right]^3$ grid points to describe 1 large eddy

$\approx 10^{11}$ grid points !!

(For each variable!)

Unsteady nature requires $\sim Re^{1/2} \sim 10^3$ time steps to describe a single large eddy turnover

Computing costs are not only astronomically, but technology doesn't exist to handle this.

B) On the structure of the small scales

Since $\frac{t_L}{t_\eta} \sim Re^{1/2}$

a) Small scales respond rapidly to any attempts by the mean flow and large scales to order their structure

b) The size and development of large structures is determined by geometry & specific forcing conditions of the flow
(Large structures are flow specific)

a) implies at small enough length scales the dynamics will be independent of the large scale motions and forcing conditions.

This gives us hope. We will develop these ideas in detail during the class.

Tensor Notation

You know what scalars, vectors & tensors are

We will commonly use "Enstein Index" Notation

Vector, $\vec{A} = (A_1, A_2, A_3) \equiv A_i$

Index represents any component

1) "If an index appears once, it is a free index & equation holds for all possible values of the index"

eg: $A_i + B_i = C_i$ represents $\underline{3}$ equations:

$$A_1 + B_1 = C_1; \quad A_2 + B_2 = C_2; \quad A_3 + B_3 = C_3$$

Gradient of a scalar: $\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right) = \frac{\partial \phi}{\partial x_i}$

$$\vec{x} = (x_1, x_2, x_3) \text{ or } (x, y, z) = x_i$$

2) "If the same free index appears twice in one term, this indicates summation over all possible values of that index"

eg: [redacted] $\rightarrow$ Dot Product

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i = a_j b_j, \text{ etc}$$

Divergence of a vector:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \frac{\partial a_i}{\partial x_i} = \frac{\partial a_k}{\partial x_k}$$

Tensor: Rate of Strain

$$S_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Nine different components ($S_{11}, S_{12}, S_{13}, S_{21}, S_{22},$

$$S_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \text{ or } \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

if $\vec{x} = (x, y, z)$ & $\vec{V} = (u, v, w)$

"Special Tensors"

$$\left.\begin{aligned} \delta_{ij} &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}\right\} \text{9 values}$$

3 = 1
6 are 0

ε_{ijk} = 1 if ijk cyclic clockwise
-1 if ijk " counter clockwise
0 if any i, j, k are equal

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$
$$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$$

$$\varepsilon_{ijk}\delta_{jk} \equiv 0$$

$\varepsilon_{112}, \varepsilon_{232}$, etc $= 0$

Gauss Theorem
Some properties ε δ $\varepsilon_{ijk}\varepsilon_{ilm}$
Symmetric, Anti

Equations

The turbulent flow field is completely described by conservation equations of mass, momentum, and energy, supplimented with appropriate initial & boundary conditions.

We will quickly review the derivation and physical interpretation of these equations

Mass: Consider an arbitrary volume in space, V

The mass in the volume ~~[illegible]~~ is simply

$$\int_V \rho \, dV$$

The rate of change of mass is then

$$\int_V \frac{\partial}{\partial t} \rho \, dV$$

This must equal the rate at which mass is flowing into the volume

$$-\int_A \rho \vec{V} \cdot d\vec{A} = -\int_V \nabla \cdot \rho \vec{V} \, dV$$

(by Guass' Theorem)

So we have

$$\int_V \frac{\partial}{\partial t} \rho \, dV + \int_V \nabla \cdot \rho \vec{V} \, dV = 0$$

or: $\int_V \left(\frac{\partial}{\partial t}\rho + \nabla\cdot\rho\vec{V}\right) dV = 0$

Since this holds for all V, the integrand must be identically = 0

$$\boxed{\frac{\partial}{\partial t}\rho + \nabla\cdot\rho\vec{V} = 0}$$

For constant density flow (ρ = const),

$$\boxed{\nabla\cdot\vec{V} = 0}$$ — $\nabla\cdot V$ describes expansion or compression of fluid elements

<u>Momentum</u>:

This is just Newton's 2nd Law ($\vec{F} = m\vec{a}$), but now applied to an arbitrary volume in space

You should all recall your momentum theorem:

"Rate of change of momentum in an arbitrary volume of fluid = net flux of momentum through the surfaces + net forces acting on volume"

Forces consist of both body & surface forces

Surface forces 1st: These guys result from existence of a stress tensor. For a Newtonian fluid:

$$\sigma_{ij} = -p\delta_{ij} + \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\underbrace{\nabla\cdot\vec{V}}_{\frac{\partial u_k}{\partial x_k}}\delta_{ij}\right]$$

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

For incompressible flow:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \leftarrow \text{rate of strain tensor}$$

Surface force in x-direction acting on $d\vec{A} = \sigma_x \cdot d\vec{A}$

This is pretty ambiguous

In Einstein Index ; $df_{s_i} = \sigma_{ij}\, dA_j$ (sum over j)

Body Forces: Many - consider gravity

$$df_{b_i} = g_i\, dV$$

Putting all together gives

$$\int_V \frac{\partial}{\partial t}\rho V_i\, dV + \int_S (\rho V_i) V_j\, dA_j = \int_V g_i\, dV + \int_S \sigma_{ij}\, dA_j$$

Applying Guass' Theorem & recognizing arbitrary volume gives

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x_j} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

For an incompressible fluid with constant μ

$$\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{V} + \vec{g}$$

or $$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 V_i}{\partial x_j \partial x_j} + g_i$$

Observations:

With appropriate IC's & BC's these equations provide a complete description of the turbulent flow field for incompressible fluid (Need an energy equation for compressible reacting flows)

Coupled, Nonlinear equations $\Rightarrow$ exact solutions for only simplest flows

Only nonlinearity is in convective terms

$$\left(\frac{\partial v_i v_j}{\partial x_j} \right)$$

Without convection equations describe viscous damping

$\Rightarrow$ Nonlinear Convective term provides mechanism for feeding energy to various length scales

Turbulence is nonlinear & we must describe nonlinear interactions

(Vortex stretching)

[redacted]

Fluid Deformation

Deformation rate tensor $\equiv \frac{\partial u_i}{\partial x_j}$

Describes relative motion of 2 adjacent fluid elements

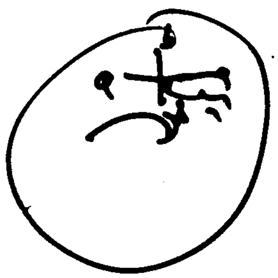
Convenent to split up as follows

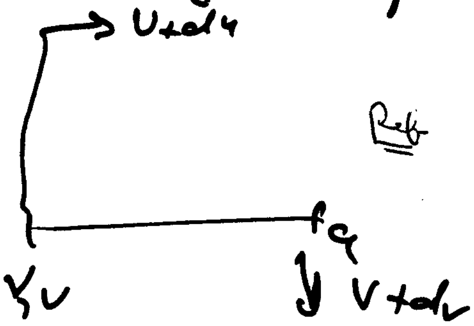
$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

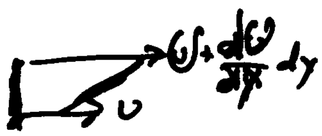
$$= \qquad S_{ij} \qquad + \qquad R_{ij}$$

S_{ij} represents pure straining motion "rate of strain" tensor

R_{ij} describes rigid body rotation



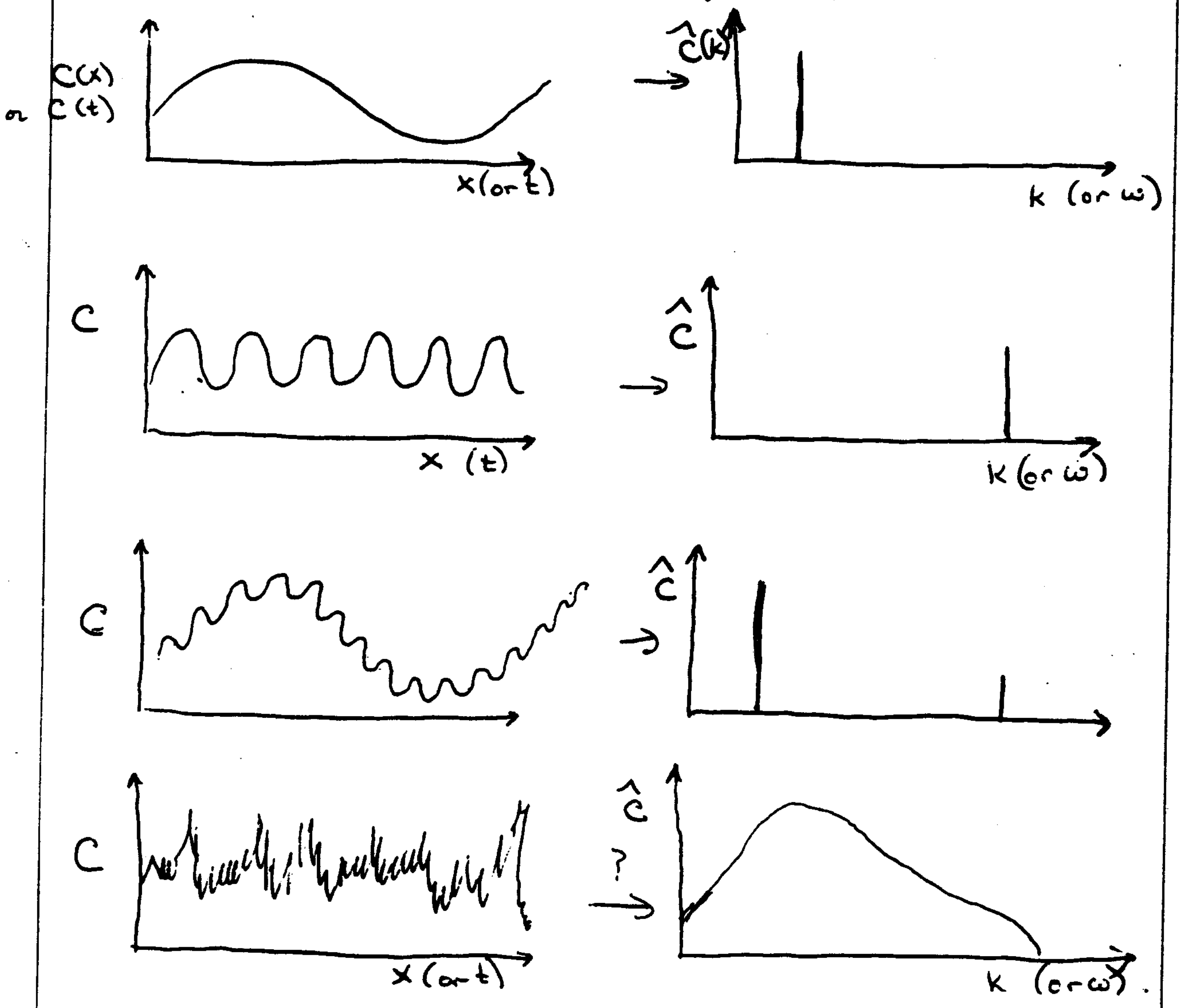




Fourier Transform

Physical Space $(\vec{x})$ or (t) $\longrightarrow$ Wave number space $(\vec{k})$

Want to describe (or decompose) a signal into its component wavelengths (or frequencies)



Any signal can be decomposed into a superposition of wavelengths & frequencies. In 1-D:

$$V(x,t) = \frac{2\pi}{L} \sum_{k=-\infty}^{\infty} \hat{V}(k,t)\, e^{ikx}$$

$$= \frac{2\pi}{L} \sum_{k=-\infty}^{\infty} \hat{V}(k,t)\left(\cos kx + i \sin kx\right)$$

$k=0$ (mean)

$k=1$ lowest wavenumber $\rightarrow$ (longest wavelength or lowest frequency

Features:

a) If $V(x)$ is real, $\hat{V}(k) = \text{c.c. } \hat{V}(-k)$

c.c. = complex conjugate

if $\hat{V} = A + iB$ then $\text{c.c. } \hat{V} = A - iB$

b) If $V(x)$ is even, then $\hat{V}(k)$ is real number (imaginary part is 0)

because cos is even

$$\cos(x) = \cos(-x)$$

c) If $V(x)$ is odd, then $\hat{V}(k)$ is an imaginary number (no real part) because sin is odd

$$\sin(x) = -\sin(-x)$$

Show b & c assuming a)

Start with $V(x) = \sum_{k=-\infty}^{\infty} \hat{V}(k)\, e^{ikx}$ ✶

Then use $e^{ikx} = \cos kx + i \sin kx$, plug into ✶, and show imaginary or real parts of $\hat{V}(k)$ must be 0 for even or odd functions.

Incompressible Constant density flow:

mass
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

or
$$\frac{\partial u_i}{\partial x_i} = 0$$

momentum;
$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

or
$$\frac{\partial u_j}{\partial t} + \frac{\partial u_j u_i}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

Fourier Transforms;

Some uses in fluid mechanics

A) Numerical: Changes differential operators into algebraic expressions

B) Physical Interpretation: Gives a direct indication of length scale information

Discrete Fourier Transform:

$$V(\vec{x},t) = \frac{2\pi}{L} \sum_{\substack{k_1, k_2, k_3 = -\infty \\ k_x, k_y, k_z}}^{\infty} \hat{V}(\vec{k},t)\, e^{i(\vec{k}\cdot\vec{x})}$$

$\sum \sum \sum_{k_x = -}$

Defined on doman size L

Inverse Transform:

$$\hat{V}(\vec{k},t) = \sum_{x_1, x_2 x_3 = -L/2}^{L/2} V(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x})}$$

Fourier Integral Transform

$$\hat{V}(\vec{x},t) = \iiint v(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x})}\, dx$$

Derivatives: simply differentiate term by term:

$$\frac{\partial V(\vec{x},t)}{\partial x_1} = \left(\frac{2\pi}{L}\right)^3 \sum_{k_1,k_2,k_3=-\infty}^{\infty} i k_1 \hat{V}(\vec{k},t)\, e^{i(\vec{k}\cdot\vec{x})}$$

$$\text{F.T.}\left(\frac{\partial U(x)}{\partial x}\right) = i k_x \hat{U}(k)$$

Fourier transform of governing Equations

Mass $\quad \vec{k}\cdot\hat{V}(\vec{k}) = 0$

This represents a separate equation for each wavenumber vector $\vec{k}$

Momentum:

Viscous term: $\quad \frac{\partial^2 U_j}{\partial x_i \partial x_i} \rightarrow -(k_1^2+k_2^2+k_3^2)\,\hat{U}_j(\vec{k})$

$\quad$ or $-k^2 \hat{U}_j(\vec{k})$

Pressure: $\quad \frac{\partial p}{\partial x_j} \rightarrow i k_j \hat{p}(\vec{k})$

$\vec{k} = \{k_j\} = (k_x, k_y, k_z)$

Non Linear Terms:

Let $u_i(\vec{x})\,u_j(\vec{x}) = W(x)$

then $W(x) = \sum_{|k|\leq\infty} \hat{w}(k)\, e^{i(\vec{k}\cdot\vec{x})}$

$$= \left[\sum_{|k|<\infty} \hat{U}_i(\vec{k})\, e^{i(\vec{k}\cdot\vec{x})}\right]\left[\hat{U}_j(\vec{k})\, e^{i(\vec{k}\cdot\vec{x})}\right]$$

This then shows us how to express $\hat{W}(\vec{k})$:

$$\hat{W}(\vec{k}) = \sum_{p+q=k\,|k|<\infty} \hat{u}_i(\vec{p})\,\hat{u}_j(\vec{q})$$

We can then write the Transformed N. E equation

as: $$\frac{\partial}{\partial t} u_j(\vec{k}) + i \sum_{\substack{k_1+k_2=k \\ p+q}} k_i \hat{V}_i(\underset{p}{\vec{k}_1})\,\hat{V}_j(\underset{q}{\vec{k}_2}) - i k_j \hat{p}(k) - \nu k^2\, \hat{U}_j(\vec{k}) \quad ☆$$

Can express differently

Take $\vec{k}\cdot ☆$

$$\vec{k}\cdot\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{V}_j(\vec{k}) = -i \sum_{k_1+k_2=k} k_i\, \hat{V}_i(\vec{k}_1)\,\hat{V}_j(k_2) - i k_j \hat{p}(k)$$

Left hand side $=0$ by incompressibility condition

So $$-ik_j \sum_{k_1+k_2=k} k_i \hat{V}_i(\vec{k}_1)\hat{V}_j(\vec{k}_2) = ik_jk_j\hat{p}(\vec{k})$$

Now $k_jk_j = k^2$

$$\hat{p}(k) = -\frac{k_\alpha}{k^2}\sum_{k_1+k_2=k} k_i\hat{V}_i(k)\hat{V}_\alpha(\vec{k}_2)$$

Expression for Fourier Transform of P in terms of velocity

So Eliminating p:

j: free

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\hat{V}_j(\vec{k}) = -i\sum_{k_1+k_2=k} k_i\hat{V}_i(\vec{k}_1)\hat{V}_j(\vec{k}_2)$$

j: free

$$+\, ik_j\frac{k_\alpha}{k^2}\sum_{k_1+k_2=k} k_i\hat{V}_i(k)\hat{V}_\alpha(k_2)$$

$$= -ik_\alpha P_{ij}\sum_{k_1+k_2=k}\hat{V}_i(\vec{k}_1)\hat{V}_\alpha(\vec{k}_2)$$

where $P_{ij}(\vec{k}) = \delta_{ij} - \frac{k_ik_j}{k^2}$

Intro to Scale Analysis

Consider heat transport in a room of dimension L
If no fluid motion, heat transported by molecular diffusion:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

At $t=0$, heater is turned on on one side of room

Q: How long for temp rise to be felt on far wall

Use scale analysis

$$\frac{\partial T}{\partial t} \sim \frac{\Delta T}{t_m}$$

$$\frac{\partial^2 T}{\partial x^2} \sim \frac{\Delta T}{L^2}$$

$$\Rightarrow \frac{\Delta T}{t_m} \sim \alpha \frac{\Delta T}{L^2} \quad \text{or} \quad \boxed{t_m \sim L^2/\alpha}$$

Also, dimensionally, this is the only relevent time scale you can form based on given info. Must be careful to define region for scale analysis!

Now turbulent velocity fluctuations characterized by u' are present

$$\frac{\partial T}{\partial t} + \frac{\partial uT}{\partial x} = 0$$

Heat Transported by velocity

Scale analysis gives:

$$\frac{\Delta T}{t_t} \sim \frac{u' \Delta T}{L}$$

or $t_t \sim \frac{L}{u'}$ Need an estimate for u'

Based on reasonable assumptions —

Buoyant acceleration $g \frac{\Delta T}{T} \sim .3 \text{ m/sec}^2$

if $\Delta T = 10K$: But throughout room, get damping, ~~guess~~ Acceleration near heater give $v' \sim 30$ cm/sec. Pick u' throughout room of 5 cm/sec

— we get $t_t \sim 2$ min for $L = 5$ m

& $t_m \sim 100$ hr " "

$$\frac{t_t}{t_m} \sim \frac{L}{u} \frac{\alpha}{L^2} = \frac{\alpha}{UL} = \frac{\nu}{UL} \quad \text{if Prandtl \#} = \frac{\nu}{\alpha} = 1$$

= Re another interpretation of Re!

So, if Length scale is imposed, Re represents ratio of turbulent to diffusive time scales

If time scale is imposed, $Re_t^{1/2}$ represents ratio of turbulent Length scale to diffusion length scale (Show this)

Turbulent Diffusivity

Laminar Flow, no convection

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x_i \partial x_i}$$

Using dimensional scale analysis:

$$\frac{\Delta c}{\Delta t} \sim \frac{D\,\Delta c}{\Delta L^2} \Rightarrow t_m \sim L^2/D \quad \star$$

If turbulent velocity fluctuations are present:

$$\frac{\partial c}{\partial t} + \frac{\partial u_i c}{\partial x_i} = D\frac{\partial^2 c}{\partial x_i \partial x_i} \quad \star\star$$

scaling gives:

$$\frac{\partial u_i c}{\partial x_i} \sim \frac{u\,\Delta c}{L}$$

$$D\frac{\partial^2 c}{\partial x_i \partial x_i} \sim \frac{D\,\Delta c}{L^2}$$

$\Rightarrow \frac{\text{convection}}{\text{diffusion}} \sim \frac{uL}{D}$; now if $\frac{uL}{D}$ is large (it usually is) convection dominates the scale analysis of $\star\star \Rightarrow t_T \sim \frac{L}{u} \quad \star\star\star$

↑ turbulence time scale

Define a turbulent diffusivity such that t_T can be expressed analagously to $\star$

$$t_T \sim L^2/D_T \quad \star\star\star\star$$

Equating $\star\star\star$ with $\star\star\star\star$ gives

$$D_T \sim uL$$

Another view point

Diffusion coefficient : Definition

$$D = \frac{\text{flux}}{\text{gradient}}$$

i.e., diffussion flux is driven by local gradients

This is clear in applications where motions are much less than over which regions the bulk properties vary (Brownian Motion)

What about turbulence?

Here fluid elements can experience motions over distances on the order of that which bulk properties vary.

How do we then define a turbulent diffusion coef?

1-D brownian motion

$$p(x,n)dx = (2\pi n l^2)^{-1/2} \exp\left[\frac{-x^2}{2nl^2}\right] dx$$

$n = Kt$

$n = \#$ of steps

$l =$ length of step

Now consider a concentration field, c

We can express

$$c(x,t) = c_0\, p(x,t) \qquad \star$$

where $c_0 = c(0,0)$

(This describes diffusion from a point source)

But what we are describing satisfies the diffusion equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \qquad \star\star$$

Put $\star$ into $\star\star$

If $\star$ satisfies $\star\star$, we must have $K = 2D/\ell^2$

$$\Rightarrow c = \frac{c_0}{2\sqrt{\pi D t}} \exp\left(-x^2/4Dt\right) \qquad \star\star\star$$

Now relate diffusivity to displacement of fluid particles:

take $\frac{c}{c_0}dx = p(x,t)\,dx$ = probability of finding a particle between x & $x+dx$ $\left(\text{so } p(x,t) = \frac{e^{-x^2/4Dt}}{2\sqrt{\pi D t}}\right)$ (A)

Mean square displacement given by

$$\langle x^2 \rangle = \int_0^\infty x^2\, 2p(x)\, dx \qquad B$$

so (A) in (B) gives

$$\boxed{\langle x^2 \rangle = 2Dt}$$

A similar analysis can be carried out for 2 & 3-D flows:

For 3-D $\langle x^2 \rangle = 6 Dt$

Note: No preferred direction in the random walk
Flux arise due to inhomogeneity of particle density

What does this have to do with turbulence?
Consider a flow with a bunch of particles that follow fluid pathlines exactly. The particles will experience fluctuations and their posistion will deviate from their initial condition. So define

$\langle x^2 \rangle = 6 D_T t$ ← definition of turbulent diffusivity

In turbulence, random walk analysis always gives correct answer

Flux / gradient interpretation may or <u>may not</u>

Statistics of The Turbulent Flow Field

Before we discuss the "Equilibrium Range Theories" it will be useful to formally define some terms like "homogeneous and "isotropic" turbulence. First we'll discuss some of the statistical characteristics of the flow.

Since turbulent flows are random in nature, we can expect to only describe the flow at any instant in time or any point in space within a certain probabi

Consider a random variable ϕ (this could be any scalar in a turbulent flow, or velocity component, et

Let $P_\phi(\psi)\,d\psi$ = probability that ϕ lies between ψ & $\psi + d\psi$

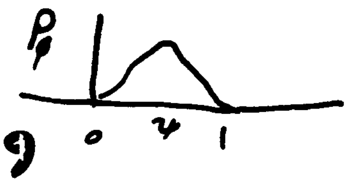
$P_\phi(\psi)$ is the probability density function

If Φ is a function of space and time, the appropriate notation indicating this is

$$P_\phi(\psi; x, t)$$

An immediate result of this is

$$\int_{-\infty}^{\infty} P_\phi(\psi; x, t)\, d\psi = 1$$



This result makes sense since all it really says is that the random variable ϕ has a probability of 1 of taking on some value

The cumulative distribution function is

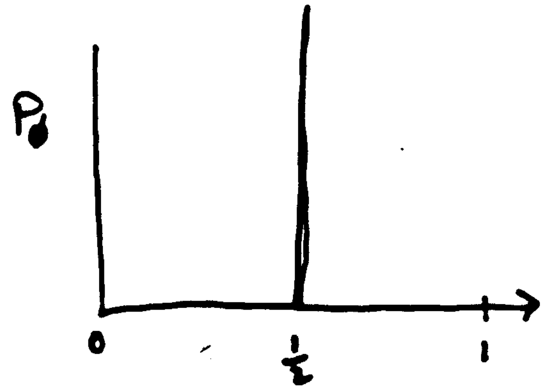
$$C(\psi) = \int_{-\infty}^{\psi} P_\phi(\psi; x,t)\, d\psi$$

The cfd is a monotonical increasing function taking on a value of 0 at $-\infty$ and 1 at $+\infty$

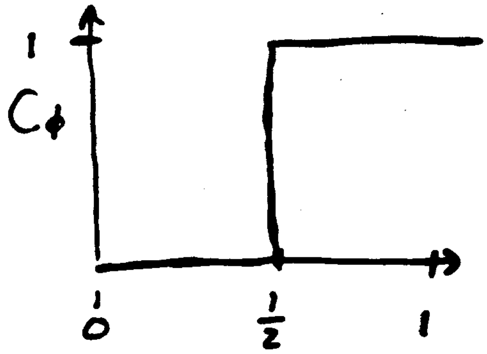
Usually, the pdf is defined in terms of the cfd

$$\underline{\underline{P_\phi = \frac{d\, C_\phi(\psi)}{d\psi}}}$$

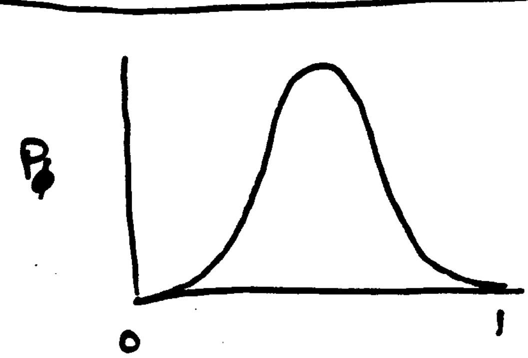
f :

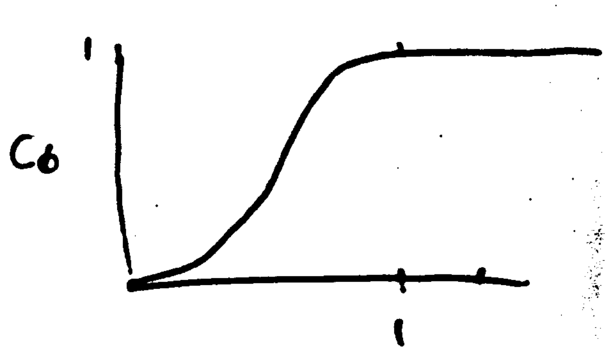


then



$P_\phi = \delta(x - \frac{1}{2})$





in words, the cfd, $C_\phi(\psi)$ is the probability that the random variable ϕ has a value less than ψ

From the pdf, all the single point statistics can be computed For example

mean: $\overline{\phi} = \int_{-\infty}^{\infty} \psi P_\phi(\psi)\, d\psi$

The second moment ~~standard~~ (variance):

$$\int_{-\infty}^{\infty} (\psi - \overline{\phi})^2 P_\phi(\psi)\, d\psi$$

All the higher moments are similarly defined

The mean can also be defined from

$$\overline{\phi} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \phi(t)\, dt$$

This gives the mean value at a point. It only makes sense if the integral is independent of t_0 (when you start sampling) and if it converges for "large enough" T.

If the previous integral does not converge it may be more useful to consider an "ensemble" or "volume" average:

$$\bar{\phi} = \lim_{Vol \to \infty} \frac{1}{Vol} \int_{Vol} \phi \, dx\,dy\,dz$$

In this case the integration is now performed over a volume at one instant in time. This makes sense only if the statistical properties do not depend on spatial position

The ensemble average is

$$\bar{\phi} = \lim_{N \to \infty} \frac{1}{N} \sum_{1}^{N} \phi$$

The summation is over a number of samples N

The summation can be over N samples, taken at the same time t for N different realizations

Multi Point Statistics

To draw conclusions about length scale information need 2 point statistics (Spatial information requires information at 2 points

The correlation tensor can give this information

$$R_{ij}(\vec{r}) = \overline{u_i(\vec{x})\, u_j(\vec{x}+\vec{r})}$$

To describe various ~~spatial~~ scales of spatial motion in a turbulent flow it is often more instructive to work with the Fourier Transform of the correlation tensor

$$\phi_{ij}(\vec{k}) = \frac{1}{2\pi}\iiint R_{ij}\exp(-i\vec{k}\cdot\vec{r})\,dr$$

$\phi_{ij}(k)$ is called the "spectrum tensor" or "spectral density" as it represents the contribution of a wavenumber k, to the value R_{ij}

Each wave number k corresponds to a physical space structure with a wavelength $2\pi/k$

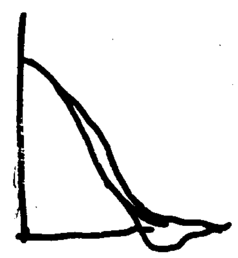
By inverse transform:

$$R_{ij}(\vec{r}) = \iiint \phi_{ij}\exp(i\vec{k}\cdot\vec{r})\,dk$$

Particularly significant: $R_{ii}(0) = u_1^2 + u_2^2 + u_3^2$

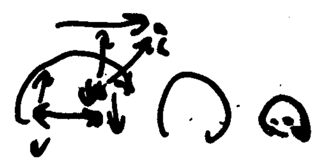
By definitions above

$$\tfrac{1}{2}R_{ii}(0) = \tfrac{1}{2}\iiint \phi_{ii}(\vec{k})\,d\vec{k}$$

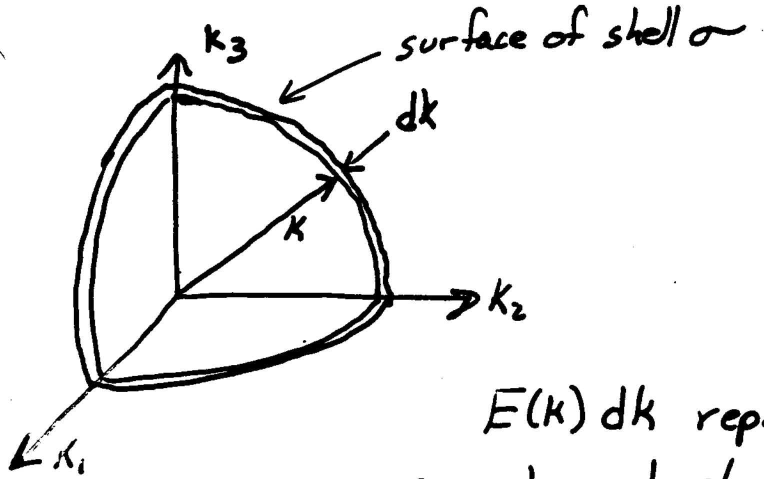


$$= \int_0^\infty \left[\tfrac{1}{2}\iint \phi_{ii}(\vec{k})\,d\sigma\right]dk$$

$$= \int_0^\infty E(k)\,dk$$



$$E(k) \equiv \tfrac{1}{2}\iint \phi_{ii}(\vec{k})\,d\sigma$$



$E(k)\,dk$ represents contribution of k.e. in a sphercal shell of thickness dk
3-D energy spectrum

$E(k)$ gives distribution of energy amoung wave numbers (which can be related to length scales)

A lot of theoretical work on turbulence is concerned with the description of energy in the wave number space & the transfer of energy amoung the various wavenumbers

Homogeneous Turbulent Flow:

Turbulent flow in which statistical properties are independent of space

Isotropic:

Statistical properties are independent of orientation, $\vec{r}$

e.g. - 2 point statistics

Statistically Steady or Stationary:

Flow whose statistics are constant in time