

Division Algorithms and Term Orderings for Symbolic Computation

Motivating Gröbner Bases and their computation

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Agenda:

- Wish to build a polynomial algebra model for hardware
- Modulo arithmetic model is versatile: can represent both *bit-level* and *word-level* constraints
- To build the algebraic/modulo arithmetic model:
 - Rings, Fields, Modulo arithmetic
 - Polynomials, Polynomial functions, Polynomial Rings
 - Ideals, Varieties, Symbolic Computing and Gröbner Bases
 - Decision procedures in verification

Consolidating the results so far....

- Ideal $J = \langle f_1, \dots, f_s \rangle \subseteq R = \mathbb{F}[x_1, \dots, x_d]$ generated by any set of polynomials f_1, \dots, f_s
- $J = \langle f_1, \dots, f_s \rangle = \{ \sum_{i=1}^s f_i \cdot h_i : h_i \in R \}$
- Many ideal generators: $J = \langle f_1, \dots, f_s \rangle = \dots = \langle g_1, \dots, g_t \rangle$
 - Given: $F = \{f_1, \dots, f_s\} \in R$
 - Gröbner basis: $G = \{g_1, \dots, g_t\}$ a canonical representation of ideal $J = \langle F \rangle = \langle G \rangle$
 - Buchberger's algorithm computes a Gröbner basis, which we will study soon
- Variety: the set of all solutions to $f_1 = \dots = f_s = 0$
- Variety depends on the ideal J , not just on f_1, \dots, f_s
- $V(f_1, \dots, f_s) = V(g_1, \dots, g_t) = V(J)$

Some facts about ideals and varieties

- When ideal $J = \langle 1 \rangle \subseteq \mathbb{F}[x_1, \dots, x_d]$, then $J = F[x_1, \dots, x_d]$
- $J = \langle 1 \rangle \iff V(J) = \emptyset$; as the polynomial $1 = 0$ has no solutions
- **Variety:** Set of ALL solutions to a given system of polynomial equations: $V(f_1, \dots, f_s)$
 - $V(x^2 + y^2 - 1) = \{\text{all points on circle : } x^2 + y^2 - 1 = 0\}$
 - $V_{\mathbb{R}}(x^2 + 1) = \emptyset$;
 - $V_{\mathbb{C}}(x^2 + 1) = \{(\pm i)\}$
- Important to analyze variety over a specific field ($V_{\mathbb{R}}$ versus $V_{\mathbb{C}}$)
- Modern algebraic geometry does not **explicitly solve for the varieties**. Rather, it reasons about the Variety by analyzing the Ideals!
 - Solving for varieties is extremely hard
 - Reasoning about their presence, absence, union/intersection is easier
 - We need to do the same for hardware verification

Formally define a variety

- Let $R = \mathbb{F}[x_1, \dots, x_d]$ be a ring, $f \in R$ be a polynomial and $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{F}^d$ be a point
- We say that f **vanishes** on \mathbf{a} when $f(\mathbf{a}) = 0$

Definition

For any ideal $J = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{F}[x_1, \dots, x_d]$, the *affine variety* of J over \mathbb{F} is:

$$V(J) = \{\mathbf{a} \in \mathbb{F}^d : \forall f \in J, f(\mathbf{a}) = 0\}.$$

Algebraically Closed Fields (ACFs)

- A field $\overline{\mathbb{F}}$ is algebraically closed, when every non-constant univariate polynomial $f \in \overline{\mathbb{F}}[x]$ has a root in $\overline{\mathbb{F}}$
- Every field is either algebraically closed, or it is contained in an algebraically closed one
- Algebraically closed fields are infinite fields
- Only over algebraically closed fields can one reason (unambiguously) about presence or absence of solutions (varieties)
 - Many famous mathematical results valid (only!) over ACFs
- Examples: \mathbb{R} is not ACF as $V_{\mathbb{R}}(x^2 + 1) = \emptyset$;
- \mathbb{C} is ACF; in fact \mathbb{C} is the algebraic closure of \mathbb{R} ($\mathbb{R} \subset \mathbb{C}$)
- Finite (Galois) fields are NOT ACF!
 - But every GF $\mathbb{F}_{p^k} \subset \overline{\mathbb{F}_{p^k}}$, where $\overline{\mathbb{F}_{p^k}}$ is the algebraic closure of \mathbb{F}_{p^k}
- So how will we reason about $V_{\mathbb{F}_{2^k}}(J)$? We will, using some funky Galois field results (Galois fields are awesome!)

There's a lot to study about Varieties, but...

- This is a good time to first think in terms of a canonical representation of ideals — i.e. a **Gröbner Bases**
- Recall:
- Given polynomials $f_1, \dots, f_s \in \mathbb{F}[x_1, \dots, x_d]$. Let $F = \{f_1, \dots, f_s\}$ be the given set of polynomials
- Then ideal $J = \langle F \rangle \subset \mathbb{F}[x_1, \dots, x_d]$
- Find another set of polynomials $G = \{g_1, \dots, g_t\} \in \mathbb{F}[x_1, \dots, x_d]$ such that:
 - $J = \langle F \rangle = \langle G \rangle$
 - $V(J) = V(\langle F \rangle) = V(\langle G \rangle)$
 - The set G has some nice properties that F does not have
 - The set G is called a **Gröbner basis of ideal J**

The power of Gröbner bases

- A Gröbner basis G can help us solve (unambiguously) many polynomial decision questions:

Ideal Membership Testing

Given ideal $J = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{F}[x_1, \dots, x_d]$, and a polynomial f , is $f \in J$?

Hilbert's Nullstellensatz: The polynomial SAT/UNSAT problem

Given ideal $J = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{F}[x_1, \dots, x_d]$, is $V(J) = \emptyset$?

The polynomial #SAT problem

Given ideal $J = \langle f_1, \dots, f_s \rangle \subseteq R = \mathbb{F}[x_1, \dots, x_d]$, is $V(J)$ infinite or finite? If finite, then $|V(J)| = ?$ [i.e. how many solutions to $V(J)$?]

Elimination ideals: help in solving polynomial equations

Generalize **triangularization** to polynomial equations

Solve the system of equations:

$$f_1 : x^2 - y - z - 1 = 0$$

$$f_2 : x - y^2 - z - 1 = 0$$

$$f_3 : x - y - z^2 - 1 = 0$$

Gröbner basis with lex term order $x > y > z$

$$g_1 : x - y - z^2 - 1 = 0$$

$$g_2 : y^2 - y - z^2 - z = 0$$

$$g_3 : 2yz^2 - z^4 - z^2 = 0$$

$$g_4 : z^6 - 4z^4 - 4z^3 - z^2 = 0$$

- Is $V(\langle G \rangle) = \emptyset$? No, because $G \neq \{1\}$
- G tells me that $V(\langle G \rangle)$ is finite!
- G is *triangular*: solve g_4 for z , then g_2, g_3 for y , and then g_1 for x

To start thinking in terms of Gröbner bases...

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- Let us study these in detail

The one variable case of $\mathbb{F}[x]$

- $f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- The terms of f are ordered according to (descending) degrees
- $\deg(f) = n$ is the degree of f
- $lt(f) = a_n x^n$ is the **leading term** of f
- $lm(f) = x^n$ is the **leading monomial** of f [often also called the leading power of $f(lp(f))$]
- $lc(f) = a_n$ is the **leading coefficient** of f
- lt, lm, lc are the main tools of the division algorithm

Polynomial Long Division (say, in $\mathbb{Q}[x]$)

Divide f by g , get q, r s.t. $f = qg + r$, with $r = 0$ or $\deg(r) < \deg(g)$

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Divide $f = x^3 - 2x^2 + 2x + 8$ by $g = 2x^2 + 3x + 1$

Division Algorithm is so Simple...

Inputs: $f, g \in \mathbb{F}[x], g \neq 0$

Outputs: q, r s.t. $f = qg + r$ with $r = 0$ or $\deg(r) < \deg(g)$

1: $q \leftarrow 0; r \leftarrow f$

2: **while** ($r \neq 0$ AND $\deg(g) \leq \deg(r)$) **do**

3: $q \leftarrow q + \frac{lt(r)}{lt(g)}$

4: $r \leftarrow r - \frac{lt(r)}{lt(g)} \cdot g$

5: **end while**

6: return q, r ;

Algorithm 1: Univariate Division of f by g

Run the algorithm on the previous example

Does this algorithm run on $\mathbb{Z}_p[x]$ as is? Say, over $\mathbb{Z}_{11}[x]$ for the previous example? What about over $\mathbb{Z}_8[x]$?

- Remember: Division is modeled as cancellation of leading terms ($lt(f)$) by leading terms ($lt(g)$)
- For $r = f - \frac{lt(f)}{lt(g)}g = f - \frac{lc(f)}{lc(g)} \cdot \frac{lm(f)}{lm(g)} \cdot g$
- Requires computation of inverse of $lc(g)$
- This division algorithm works over fields $\mathbb{F} = \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_p, \mathbb{F}_{2^k}$, etc.
- This division algorithm does not always work over $\mathbb{Z}, \mathbb{Z}_n, n \neq p$.

- Let $f = x$; $f_1 = x^2$; $f_2 = x^2 - x$ in $\mathbb{Q}[x]$

Application to Ideal Membership Test

- Let $f = x$; $f_1 = x^2$; $f_2 = x^2 - x$ in $\mathbb{Q}[x]$
- Is $f \in J = \langle f_1, f_2 \rangle$?

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Application to Ideal Membership Test

- Let $f = x$; $f_1 = x^2$; $f_2 = x^2 - x$ in $\mathbb{Q}[x]$
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- $f \xrightarrow{f_1} f \xrightarrow{f_2} f \neq 0$
- $f \xrightarrow{f_2} f \xrightarrow{f_1} f \neq 0$
- What's happening?
- $F = \{f_1, f_2\}$ is not a Gröbner basis of J
- Cannot decide ideal membership without Gröbner basis!

- When \mathbb{F} is a field, **Every ideal** J of $\mathbb{F}[x]$ is generated by only one element (polynomial).
 - These rings $\mathbb{F}[x]$ are **principal ideal domains (PID)**
 - *E.g.* $\mathbb{Z}_p[x] = \text{PID}$, but multivariate rings are not PIDs (*e.g.* $\mathbb{Z}_p[x_1, x_2] \neq \text{PID}$)
 - Ideal of vanishing polynomials is a good example: $\langle x^p - x \rangle$ versus $\langle x_1^p - x_1, x_2^p - x_2 \rangle$
- Gröbner Basis of $\{f_1, f_2\} = \text{GCD}(f_1, f_2)$
- Gröbner Basis of $\{f_1, \dots, f_s\} = \text{GCD}(f_1, \text{GCD}(f_2, \dots, f_s))$
- The Euclidean Algorithm computes the GCD of two polynomials
- The algorithm is given in any math textbook, and can also be found on wikipedia (Internet)
- Homework assignment for you..... Euclidean algorithm
- Univariate rings are of not much use in hardware verification

- Divide $f = y^2x + 4yx - 3x^2$ by $g = 2y + x + 1$

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- What are $lt(f), lt(g)$?
- We need to figure out how to order the terms of f, g

Monomial (Term) Orderings

Power product: $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $\alpha_i \in \mathbb{Z}_{\geq 0}$.

For simplicity: $x_1^{\alpha_1} \dots x_d^{\alpha_d} = \mathbf{x}^\alpha$, $\alpha \in \mathbb{Z}_{\geq 0}^d$.

- Term = $a \cdot \mathbf{x}^\alpha$ = coeff. times a power product
- $\mathbb{T}^d = \{\mathbf{x}^\alpha : \alpha \in \mathbb{Z}_{\geq 0}^d\}$ is the set of all power products
- A multivariate polynomial is a sum of terms

Impose a Monomial Ordering on $\mathbb{F}[x_1, \dots, x_d]$

A total order $<$ on \mathbb{T}^d , and it should be a well-ordering:

- Total order: One and only one of the following should be true:
 $x^\alpha > x^\beta$ or $x^\alpha = x^\beta$ or $x^\alpha < x^\beta$.
- $1 < x^\alpha$, $\forall x^\alpha$ ($x^\alpha \neq 1$)
- $x^\alpha < x^\beta \implies x^\alpha \cdot x^\gamma < x^\beta \cdot x^\gamma$.

Definition (LEX)

Lexicographic order: Let $x_1 > x_2 > \dots > x_d$ lexicographically. Also let $\alpha = (\alpha_1, \dots, \alpha_d)$; $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_{\geq 0}^d$. Then we have:

$$x^\alpha < x^\beta \iff \left\{ \begin{array}{l} \text{Starting from the left, the first co-ordinates of } \alpha_i, \beta_i \\ \text{that are different satisfy } \alpha_i < \beta_i \end{array} \right.$$

For 2-variables: $1 < x_2 < x_2^2 < \dots < x_1 < \dots < x_2 x_1 < \dots < x_1^2 < \dots$

Definition (DEGLEX)

Degree Lexicographic order: Let $x_1 > x_2 > \dots > x_d$ lexicographically. Also let $\alpha = (\alpha_1, \dots, \alpha_d); \beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_{\geq 0}^d$. Then we have:

$$x^\alpha < x^\beta \iff \begin{cases} \sum_{i=1}^d \alpha_i < \sum_{i=1}^d \beta_i \\ \sum_{i=1}^d \alpha_i = \sum_{i=1}^d \beta_i \text{ AND } x^\alpha < x^\beta \end{cases} \quad \text{OR} \\ \text{w.r.t. LEX order}$$

Definition (DEGREVLEX)

Degree Reverse Lexicographic order: Let $x_1 > x_2 > \dots > x_d$ lexicographically. Also let $\alpha = (\alpha_1, \dots, \alpha_d); \beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_{\geq 0}^d$. Then we have:

$$x^\alpha < x^\beta \iff \begin{cases} \sum_{i=1}^d \alpha_i < \sum_{i=1}^d \beta_i \text{ or} \\ \sum_{i=1}^d \alpha_i = \sum_{i=1}^d \beta_i \text{ AND the first co-ordinates} \\ \alpha_i, \beta_i \text{ from the RIGHT, which are different, satisfy } \alpha_i > \beta_i \end{cases}$$

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Always fix a term order over a ring, and stick to it!

$$f = c_1X_1 + c_2X_2 + \cdots + c_tX_t \text{ implies } X_1 > \cdots > X_t$$

Divide $f = y^2x + 4yx - 3x^2$ by $g = 2y + x + 1$ with DEGLEX $y > x$ in $\mathbb{Q}[x, y]$

Divide $f = y^2x + 4yx - 3x^2$ by $g = 2y + x + 1$ with DEGLEX $y > x$ in $\mathbb{Q}[x, y]$

Solved on the board in the classroom

Divide f by g : denoted $f \xrightarrow{g} h$, where $h = f - \frac{X}{\text{lt}(g)}g$. Here, X may not be the leading term.

Definition

Let $f, f_1, \dots, f_s, h \in \mathbb{F}[x_1, \dots, x_n]$, $f_i \neq 0$; $F = \{f_1, \dots, f_s\}$. Then f reduces to h modulo F :

$$f \xrightarrow{F}_+ h$$

if and only if there exists a sequence of indices $i_1, i_2, \dots, i_t \in \{1, \dots, s\}$ and a sequence of polynomials $h_1, \dots, h_{t-1} \in k[x_1, \dots, x_n]$ such that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \dots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h$$

$f_1 = yx - y, f_2 = y^2 - x, f = y^2x$; DEGLEX $y > x$ in $\mathbb{Q}[x]$. Divide $f \xrightarrow{f_1, f_2}_+ h$:

- To divide f by $F = \{f_1, \dots, f_3\}$ (say)
- Impose term order on the ring
- Impose the given order on polynomials of F : $f_1 > f_2 > f_3$
- Divide f by f_1 first:
 - Analyze terms of $f = c_1X_1 + c_2X_2 + \dots + c_tX_t$ in order
 - Does $lt(f_1) \mid c_1X_1$? If so, divide (or cancel $lt(f)$), update f , and check if $lt(f_1) \mid$ the new $lt(f)$ (in updated f)?
 - Otherwise, does $lt(f_2) \mid c_1X_1$? And so on...
- If $lt(f)$ is not divisible by any $lt(f_i)$, then move $lt(f)$ into the remainder ($r = r + lt(f)$), and update f ($f = f - lt(f)$)
- Repeat... [See the algorithm in the next slides]

Definition

If $f \xrightarrow{F}_+ r$, then no term in r is divisible by $\text{LT}(f_i)$, $\forall f_i \in F$. Then r is reduced w.r.t. F and it is called the remainder.

Definition

Let $f, f_1, \dots, f_s, r \in \mathbb{F}[x_1, \dots, x_n]$, $f_i \neq 0$; $F = \{f_1, \dots, f_s\}$. Then f reduces to r modulo F :

$$f \xrightarrow{F}_+ r$$

then

$$f = u_1 f_1 + \dots + u_s f_s + r$$

and we have that:

- r is reduced w.r.t. F
- $u_1, \dots, u_s \in \mathbb{F}[x_1, \dots, x_n]$
- $\text{LP}(f) = \text{MAX}(\text{LP}(f_1)\text{LP}(u_1), \dots, \text{LP}(f_s)\text{LP}(u_s), \text{LP}(r))$

Multivariate Division Algorithm

Inputs: $f, f_1, \dots, f_s \in \mathbb{F}[x_1, \dots, x_n], f_i \neq 0$

Outputs: u_1, \dots, u_s, r s.t. $f = \sum f_i u_i + r$ where r is reduced w.r.t. $F = \{f_1, \dots, f_s\}$ and $\max(\text{lp}(u_1)\text{lp}(f_1), \dots, \text{lp}(u_s)\text{lp}(f_s), \text{lp}(r)) = \text{lp}(f)$

1: $u_i \leftarrow 0; r \leftarrow f, h \leftarrow f$

2: **while** ($h \neq 0$) **do**

3: **if** $\exists i$ s.t. $\text{lm}(f_i) \mid \text{lm}(h)$ **then**

4: choose i least s.t. $\text{lm}(f_i) \mid \text{lm}(h)$

5: $u_i = u_i + \frac{\text{lt}(h)}{\text{lt}(f_i)}$

6: $h = h - \frac{\text{lt}(h)}{\text{lt}(f_i)} f_i$

7: **else**

8: $r = r + \text{lt}(h)$

9: $h = h - \text{lt}(h)$

10: **end if**

11: **end while**

Algorithm 2: Multivariate Division of f by $F = \{f_1, \dots, f_s\}$

Let $F = \{f_1, \dots, f_s\}$; $J = \langle f_1, \dots, f_s \rangle$ and let $f \in J$. Then we should be able to represent $f = u_1 f_1 + \dots + u_s f_s + r$ where $r = 0$. If we were to divide f by $F = \{f_1, \dots, f_s\}$, then we will obtain an intermediate remainder (say, h) after every one-step reduction. The leading term of every such remainder ($\text{LT}(h)$) should be divisible by the leading term of at least one of the polynomials in F . Only then we will have $r = 0$.

Definition

Let $F = \{f_1, \dots, f_s\}$; $G = \{g_1, \dots, g_t\}$;
 $J = \langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle$. Then G is a **Gröbner Basis** of J



$$\forall f \in J \ (f \neq 0), \quad \exists i : \text{LM}(g_i) \mid \text{lm}(f)$$

Definition

$$G = \{g_1, \dots, g_t\} = GB(J) \iff \forall f \in J, \exists g_i \text{ s.t. } \text{Im}(g_i) \mid \text{Im}(f)$$

Definition

$$G = GB(J) \iff \forall f \in J, f \xrightarrow{g_1, g_2, \dots, g_t} + 0$$

Implies a “decision procedure” for ideal membership

Buchberger's Algorithm

INPUT : $F = \{f_1, \dots, f_s\}$

OUTPUT : $G = \{g_1, \dots, g_t\}$

$G := F;$

REPEAT

$G' := G$

 For each pair $\{f, g\}, f \neq g$ in G' DO

$S(f, g) \xrightarrow{G'} r$

 IF $r \neq 0$ THEN $G := G \cup \{r\}$

UNTIL $G = G'$

$$S(f, g) = \frac{L}{lt(f)} \cdot f - \frac{L}{lt(g)} \cdot g$$

$L = \text{LCM}(lm(f), lm(g)), \quad lm(f)$: leading monomial of f