Nov 1. Recap.

Univariate Division

$$f = LT(f) + Tail(f)$$

$$g = LT(g) + Tail(g)$$

$$f \xrightarrow{g} r \ : \ r = f - \frac{LT(f)}{LT(g)} \cdot g$$

dividend

quotient

divisor

1-step remainder

$$f \xrightarrow{g}\mathrel{\mkern-14mu}+ \; r$$

final remainder

Multivariate polynomials

→ need term ordering.

Lex, deglex, degrevlex

<u>Ex</u>: $f = y^2x + 4yx - 3x^2$

$g = 2y + x + 1$ $\quad \mathbb{Q}[x,y]$

deglex $y > x$

$$r = f - \frac{c \cdot X}{LT(g)} \cdot g$$

$c \cdot X$ = 1st monomial in f that can be cancelled by $LT(g)$.

$$= f - \frac{y^2x}{2y} \cdot g$$

$$= -\frac{1}{2}yx^2 + \frac{7}{2}yx - 3x^2$$

Poly reduction via division

Given: f, $F = \{f_1, \cdots, f_s\}$.

$$f \xrightarrow{F=\{f_1 \cdots f_s\}}_{+} r$$

1. Take $LT(f)$.
2. Divisible by $LT(f_1)$? If so,

$$r = f - \frac{LT(f)}{LT(f_1)} \cdot f_1$$

3. Continue. r = new f.

   Is $LT(r)$ divisible by $LT(f_1)$?

   If not, is $LT(r) \div LT(f_2)$?

   $LT(f_3)$?

4. If $LT(r)$ not divisible by $LT(f_i)$ $\forall i$, then $LT(r)$ = remainder.

Ex lex $x > y$ $\mathbb{Q}[x,y]$

$f = xy^2 + 1$, $I = \langle f_1, f_2 \rangle$

$f_1 = xy + 1$, $f_2 = y + 1$

1. $LT(f_1) \mid LT(f)$? ✓

$$r = f - \frac{LT(f)}{LT(f_1)} \cdot f_1$$

$$= (xy^2+1) - \left[\frac{xy^2}{xy}\right](xy+1)$$

$$= xy^2 + 1 - (y)(xy+1)$$

$$= \cancel{xy^2} + 1 - \cancel{xy^2} - y$$

$$= -y + 1 = \underline{r}$$

$r = -y+1, \quad f_1 = xy^2+1, \quad f_2 = y+1$

$\underline{\underline{2}}$ $\quad new(f) = r = -y+1$

Is $LT(f) \div LT(f_1)$?

$-y \div xy^2$? No.

Pick f_2 now.

Is $LT(f) \div LT(f_2)$?

$-y \div y$? ✓

$$r = f - \frac{LT(f)}{LT(f_2)} \cdot f_2$$

$$= (-y+1) - (-1)(y+1)$$

$$= \cancel{-y}+1 + \cancel{y}+1 = \underline{\underline{2}} = r$$

Step 3. $r = 2 = new(f)$

Is $LT(f) \div LT(f_1)$? ×

Is $LT(f) \div LT(f_2)$ ×

move 2 into the remainder.

$r = 2, \quad new f = f - 2$
$= 2 - 2 = 0$

no more terms to cancel.

So. $f \xrightarrow{f_1\, f_2}_{+} 2$

Motivate Groebner basis by means of ideal membership testing:

$J = \langle f_1, \ldots, f_s \rangle$ & another poly f: Given.

Assume $f \in J$.

$$f = u_1 f_1 + u_2 f_2 + \cdots + u_s f_s + \underbrace{\boxed{r}}_{=0}$$

1. 1-step reduction.

$$f \xrightarrow{f_1} r_1 = f - u_1 \cdot f_1$$

$f_1 \in J$, $f \in J$, $u_1 f_1 \in J$ so $r_1 \in J$.

r_1 should have a $LT(r_1)$

$LT(f_i)$, (some f_i) should divide $LT(r_1)$

$$r_2 = r_1 - \frac{LT(r_1)}{LT(f_i)} \cdot f_i$$

$r_2 \in J$ ✓

$LT(r_2)$ should also be cancelled!

.... ultimately, all terms should cancel ...

Example. $f = x$. $f_1 = x^2$, $f_2 = x^2 - x$

$f = f_1 - f_2$. $I = \langle f_1, f_2 \rangle$

So $f \in I$.

But. $LT(f_1) \nmid LT(f)$ $x^2 \nmid x$

$LT(f_2) \nmid LT(f)$ $x^2 \nmid x$.

→ Division, by itself, cannot decide ideal membership. Why?

→ Because, I does not have "all the requisite leading terms".

So How to obtain these missing leading terms in the ideal?

Think Gaussian elimination.

$f_1: \ 2x + 3y = 4 \qquad 3f_1: \ \cancel{6x} + 9y = 12$

$f_2: \ 3x + 2y = 1 \qquad -2f_2: \ \underline{-\cancel{6x} - 4y = -2}$

new leading term $\longrightarrow$ $5y = 10$

$$r = \boxed{3}\, f_1 - \boxed{2}\, f_2$$

$$= 5y - 10$$

$\uparrow$ new leading term.

$$I = \langle f_1, f_2 \rangle = \langle f_1, f_2, r \rangle$$

given f and $I = \langle f_1, f_2, f_3 \rangle$.

Is f in I?

Compute $G = GB(I)$

$= \{g_1, \ldots, g_t\}$

$$f \xrightarrow{g_1, g_2, \ldots\ldots g_t} {}_+ 0 \,?$$

Ideal membership test!