Ideals, Varieties and Symbolic Computation

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Lectures: Sept 25, 2017 onwards

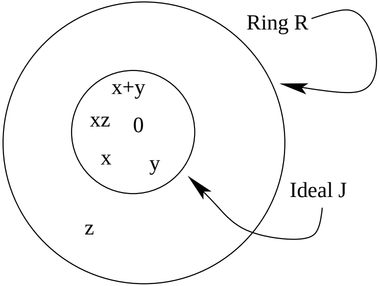
Agenda:

- Wish to build a polynomial algebra model for hardware
- Modulo arithmetic model is versatile: can represent both *bit-level* and *word-level* constraints
- To build the algebraic/modulo arithmetic model:
  - Rings, Fields, Modulo arithmetic
  - Polynomials, Polynomial functions, Polynomial Rings
  - Ideals, Varieties, Symbolic Computing and Gröbner Bases
  - Decision procedures in verification

Ideals in Rings

$R =$ ring, Ideal $J \subseteq R$, s.t.:

- $0 \in J$
- $\forall x, y \in J, x + y \in J$
- $\forall x \in J, z \in R, x \cdot z \in J$



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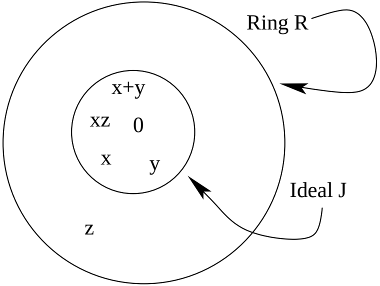

- Examples of Ideals: $R = \mathbb{Z}, J = 2\mathbb{Z}, 3\mathbb{Z}, \ldots, n\mathbb{Z}$
- Ideals versus Subrings: $\mathbb{Z} \subset \mathbb{Q}$, but $\mathbb{Z}$ not an ideal in $\mathbb{Q}$
- $1 \in$ Ring R, but 1 need not be in ideal J

Definition

Ideals of Polynomials: Let $f_1, f_2, \ldots, f_s \in R = \mathbb{F}[x_1, \ldots, x_d]$. Let

$$J = \langle f_1, f_2 \ldots, f_s \rangle = \{f_1 h_1 + f_2 h_2 + \cdots + f_s h_s : \ h_1, \ldots, h_s \in R\}$$

$J = \langle f_1, f_2 \ldots, f_s \rangle$ is an ideal generated by $f_1, \ldots, f_s$ and the polynomials are called the generators (basis) of J. [Note, h_i: arbitrary elements in R]

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Given $f_i \in J, h_i \in R$ is $f_i \cdot h_i \in J$?

Generators of Ideals

- An ideal may have many different generators
- It is possible to have:
  $J = \langle f_1, \ldots, f_s \rangle = \langle p_1, \ldots, p_l \rangle = \cdots = \langle g_1, \ldots, g_t \rangle$
- Where $f_i, p_j, g_k \in \mathbb{F}[x_1, \ldots, x_d]$ and $J \subseteq \mathbb{F}[x_1, \ldots, x_d]$
- Does there exist a Canonical representation of an ideal?
- A Gröbner Basis is a canonical representation of the ideal, with many nice properties that allow to solve many polynomial decision questions
- Buchberger's Algorithm allows to compute a Gröbner Basis
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  - Why is this important? [We'll see a little later....]

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The Ideal Membership Testing Problem

Given $R = \mathbb{F}[x_1, \ldots, x_d], f_1, \ldots, f_s, \quad f \in R$, let $J = \langle f_1, \ldots, f_s \rangle \subseteq R$. Find out whether $f \in J$?

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f = specification, J = implementation, Do an equivalence check: Is $f \in J$? [Or something like that...]

Varieties of Ideals

Given $R = \mathbb{F}[x_1, \dots, x_d], f_1, \dots, f_s, \in R$, let $J = \langle f_1, \dots, f_s \rangle \subseteq R$. The set of all solutions to:

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$J = \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$, then $V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t)$

Prove that Variety depends on the Ideal

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- Let $f \in J$

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- $f = f_1 h_1 + \cdots + f_s h_s$
- $f(\mathbf{a}) = f_1(\mathbf{a}) h_1 + \cdots + f_s(\mathbf{a}) h_s = 0$
- Extend the argument to all $f \in J$ for all $\mathbf{a} \in V(J)$, and you can show that Variety depends on the ideal $J = \langle f_1, \dots, f_s \rangle$, not just on the set of polynomials $F = \{f_1, \dots, f_s\}$

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Note $V(I_1) = V(I_2) = \{(\pm 2, \pm 1)\}$