The Boolean Satisfiability (SAT) Problem, SAT Solver Technology, and Equivalence Verification

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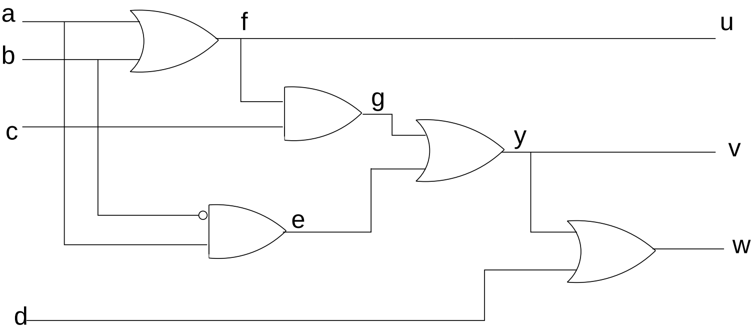
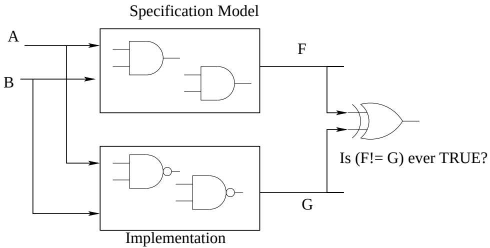
What is Boolean Satisfiability (SAT)?

- Given a Boolean formula $f(x_1, \ldots, x_n)$, find an assignment to $x_1, \ldots, x_n$ s.t. $f = 1$
- Otherwise, prove that such an assignment does not exist: problem is infeasible!
- There may be many SAT assignments: find an assignment, or enumerate all assignments (ALL-SAT)
- The formula f is given in conjunctive normal form (CNF), SAT solvers operate CNF representation of f
- Any decidable decision problem can be formulated and solved as SAT
- SAT is fundamental, has wide applications in many areas: hardware & software verification, graph theory, combinatorial optimization, artificial intelligence, VLSI design automation, cryptography/cryptanalysis, planning, scheduling, many more....

- Simulation vector generation: Given the circuit below, find an assignment to primary inputs s.t. $u = 1, v = 1, w = 0$, or prove that one does not exits
- Translate the circuit into CNF, and solve SAT



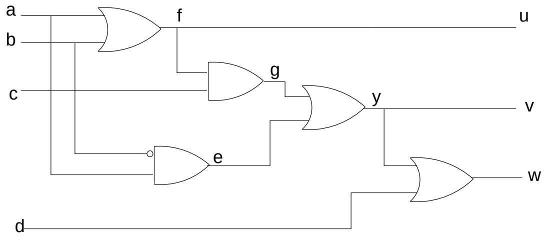
- Prove infeasibility of the miter!
  - Find an assignment to the inputs s.t. $(F \neq G) = 1$ (bug)
  - If no assignment (infeasible), circuits are equivalent
- Model checking: find an assignment s.t. a property is satisfied/falsified



SAT formulation

- A Boolean formula $f(x_1, \ldots, x_n)$ over propositional variables $x_1, \ldots, x_n \in \{0, 1\}$, using propositional connectives $\neg, \vee, \wedge$, parenthesis, and implications $\implies$, $\iff$
  - Example: $f = ((\neg x_1 \wedge x_2) \vee x_3) \wedge (\neg x_2 \vee x_3)$
- A CNF formula representation of f is:
  - a conjunction of clauses
  - each clause is a disjunction of literals
  - each literal is a variable or its negation (complement)
- Example: $f = (\neg x_1 \vee x_2)(\neg x_2 \vee x_3 \vee \neg x_4)(x_1 \vee x_2 \vee x_3 \vee \neg x_4)$
- Alternate notation $f = (x_1' + x_2)(x_2' + x_3 + x_4')(x_1 + x_2 + x_3 + x_4')$
- Any Boolean formula (circuit) can be encoded into CNF

Encode a Circuit to CNF



$$
\begin{array}{lcl}
f & = & a \vee b \\
f & \iff & a \vee b \;\; \text{(equality is a double-implication)} \\
CNF: & & (f \implies (a \vee b)) \;\wedge\; ((a \vee b) \implies f) \\
 & & (\neg f \vee (a \vee b)) \;\wedge\; (\neg(a \vee b) \vee f) \\
 & & (\neg f \vee (a \vee b)) \;\wedge\; ((\neg a \wedge \neg b) \vee f)) \;\; \text{(CNF?)} \\
 & & (\neg f \vee (a \vee b)) \;\wedge\; (\neg a \vee f)(\neg b \vee f)
\end{array}
$$

Circuit to CNF: Implication to Clauses

In general, if $f = OP(a, b)$, the CNF representation is:

- $f \iff OP(a, b)$, further simplified as:
- $(f \implies OP(a, b)) \wedge (OP(a, b) \implies f)$
- Translate implication to Boolean formula: $a \implies b$ means $(a' + b)$ is TAUTOLOGY.

- For $f = a \wedge b$, CNF: $(\neg f + a)(\neg f + b)(\neg a + \neg b + f)$
- For $f = a \oplus b$, CNF: $(\neg f + a + b)(f + \neg a + b)(f + a + \neg b)(\neg f + \neg a + \neg b)$
- For the previous circuit, we need to further constrain $u = 1, v = 1, w = 0$ to solve the simulation vector generation problem. Encode constraints $u = 1, v = 1, w = 0$ into CNF as $(u)(v)(w')$
- Conjunct ALL clauses (constraints) and invoke a SAT solver to find a solution

- In general, SAT is NP-complete. No polynomial-time algorithm exists to solve SAT (in theory).
- The restricted 2-SAT problem, where every clause contains only 2 literals, can be solved in polynomial time.
- Circuit-to-CNF: Recall, 2-input AND/OR gates need a 3-literal clause for modeling the constraint.
  - Circuit-SAT is therefore also NP-complete.
- However, modern SAT solvers are a success story in Computer Science and Engineering. Efficient heuristics and implementation tricks make SAT solvers very efficient.
- EDA gave a big impetus to SAT solving
- Many large problems can be solved very quickly by SAT solvers.
- So, how is a CNF SAT formula solved?

- An assignment can make a clause satisfied or unsatisfied
- Since $f = C_1 \wedge C_2 \wedge \cdots \wedge C_n$, try to SATISFY each clause C_i
- The first approach by Davis & Putnam [DP 1960]: based on unit clause, pure literal and resolution rules
- Later Davis, Logemann, Loveland [DLL 1962] proposed an alternative backtrack-based search algorithm
- These algorithms are now known as DPLL algorithms
- Modern solvers are highly sophisticated: conclict-driven clause learning (CDCL) and search-space pruning, among many efficient heuristics

Basic Processing for SAT solving

Satisfy a clause

A clause is satisfied if any literal is assigned to 1. E.g. for $x_2 = 0$, clause $(x_1 \vee \neg x_2 \vee \neg x_3) = 1$.

Satisfy a clause

A clause is unsatisfied if all literals are assigned to 0. E.g. the assignment of $x_1 = 0, x_2 = x_3 = 1$, makes clause $(x_1 \vee \neg x_2 \vee \neg x_3)$ unsatisfied.

Unit clause

A clause containing a single unassigned literal, and all other literals assigned to 0. E.g., the assignment $x_1 = 0, x_3 = 1$, makes $(x_1 \vee \neg x_2 \vee \neg x_3) = (0 \vee \neg x_2 \vee 0)$ a unit clause. Unit clause forces a necessary assignment $(x_2 = 0)$ for the formula to be TRUE.

- Formula f is satisfied, if all clauses are satisfied; f is unsatisfied, if at least one clause is unsatisfied.

Pure Literals

- A literal is pure if it appears only as a positive literal, or only as a negative literal.
  - $f = (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2) \wedge (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$
  - x_1, x_3 are pure literals.
- Clauses containing pure literals can be easily satisfied.
  - Assign pure literals to the values that satisfy the clauses
  - Pure literals do not cause inconsistent value assignments (or conflicts) to variables.
- Iteratively apply unit clause propagation and pure literal simplifcation on the CNF formula

- Resolution Rule: Given clauses $(x \vee \alpha)$ and $(\neg x \vee \beta)$, infer $(\alpha \vee \beta)$
  - $RES(x \vee \alpha, \neg x \vee \beta) = (\alpha \vee \beta)$
- The DP algorithm was resolution-based

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Terminate when empty clause (UNSAT) or empty formula (SAT)

Deduce SAT/UNSAT by Resolution: Example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4)$$

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(x_3)

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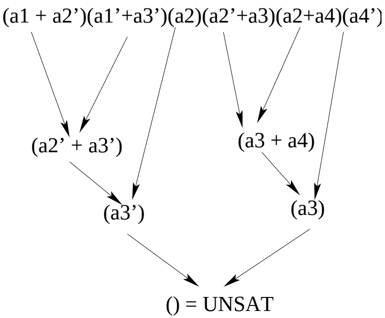
(x_3)

Satisfiable!

Resolution Proofs of UNSAT Problems



Resolution Proof

A resolution proof is a directed acyclic graph (DAG) with vertices corresponding to clauses and edges corresponding to resolution operations. Root vertices are original clauses, intermediate vertices are resolvent clauses, and the leaf vertex is an empty clause.

- The [DP 1960] approach using resolution was inefficient
- Then the [DLL 1962] was introduced:
  - Select a variable x, assign either $x = 0$ or $x = 1$ [decision assignment]
  - Simplify formula with unit propagation, pure literal rules [deduce]
  - If conflict, then backtrack [diagnose]
    - If cannot backtrack further, return UNSAT
  - If formula satisfied, return SAT
  - Otherwise, proceed with another decision

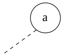
DPLL Example

$f = (a + b' + d)(a + b' + e)(b' + d' + e')(a + b + c + d)(a + b + c + d')(a + b + c' + e)(a + b + c' + e')$

DPLL Example

$f = (a + b' + d)(a + b' + e)(b' + d' + e')(a + b + c + d)(a + b + c + d')(a + b + c' + e)(a + b + c' + e')$

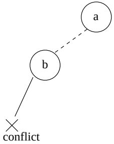
$a = 0$



DPLL Example

$f = (a + b' + d)(a + b' + e)(b' + d' + e')(a + b + c + d)(a + b + c + d')(a + b + c' + e)(a + b + c' + e')$

$a = 0$, $b = 1$, conflict, backtrack, change last decision!



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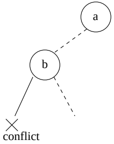
$a = 0, \ b = 0$



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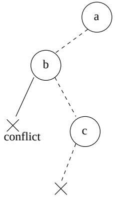
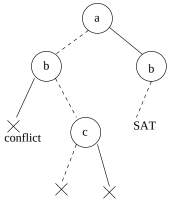
$a = 0,\ b = 0,\ c = 0,$
conflict, backtrack!



DPLL Example

$f = (a + b' + d)(a + b' + e)(b' + d' + e')(a + b + c + d)(a + b + c + d')(a + b + c' + e)(a + b + c' + e')$

$a = 1, \; b = 0$



Non-chronological Backtracking via CDCL

- Previous example shows a chronological backtrack based binary search
- Modern SAT solvers analyze decisions and conflicts to dynamically learn clauses
  - Conflict Driven Clause Learning (CDCL)
  - Solver learns more clauses, and appends them to the original CNF
  - More constraints help to prune the search
  - Results in a non-chronological backtrack-based search
  - The approach is still complete: Will find SAT, or will prove UNSAT
- There are also "incomplete" solvers, that rely on local search
  - Heuristics to guide the search, but search not exhaustive
  - May find a SAT solution if one exists, but cannot prove UNSAT
- There are also SAT pre-processors
  - Input CNF $\mathcal{F}_1$, output CNF $\mathcal{F}_2$, size($\mathcal{F}_1$) $>$ size($\mathcal{F}_2$)

Conflict-Driven Clause Learning (CDCL) solvers

- Modern CDCL-solvers: based on DPLL, but do quite a bit more
  - Learn new constraints while encountering conflicts
  - Enable non-chronological backtracking, thus pruning search-space
  - Branching heuristics: which variable to branch on ($x_i = 0$? or $x_i = 1$?)
  - Heuristics for search re-starts
  - Efficient management of clause-database: minimize learnt clauses, discard unused learnt clauses
- Concept of CDCL from [GRASP, Joao Marques-Silva and Karem Sakallah]
- Read GRASP report on class website

CDCL & Non-Chronological Backtracking [From GRASP]

$$(x_1' + x_2)(x_1' + x_3 + x_9)(x_2' + x_3' + x_4)(x_4' + x_5 + x_{10})(x_4' + x_6 + x_{11})$$
$$(x_5' + x_6')(x_1 + x_7 + x_{12}')(x_1 + x_8)(x_7' + x_8' + x_{13}')(y_1 + z_1)(y_2 + z_2)$$

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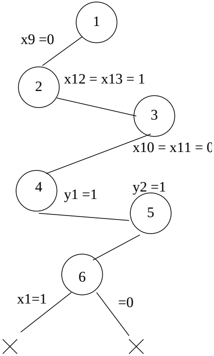

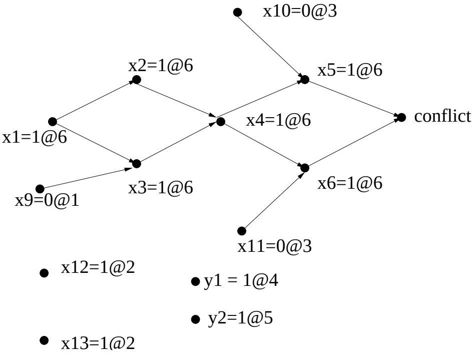

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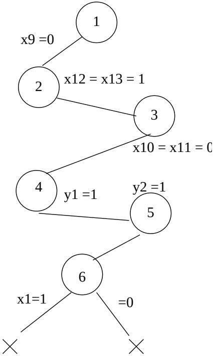

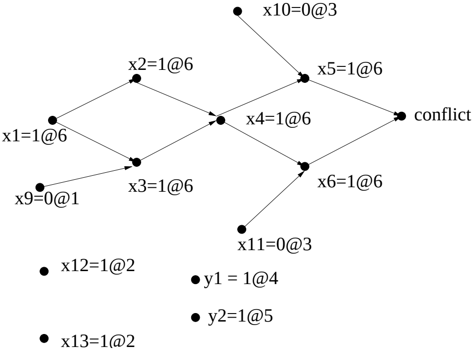

$$(x_1' + x_2)(x_1' + x_3 + x_9)(x_2' + x_3' + x_4)(x_4' + x_5 + x_{10})(x_4' + x_6 + x_{11})$$
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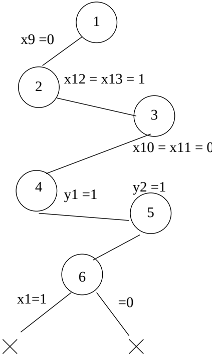

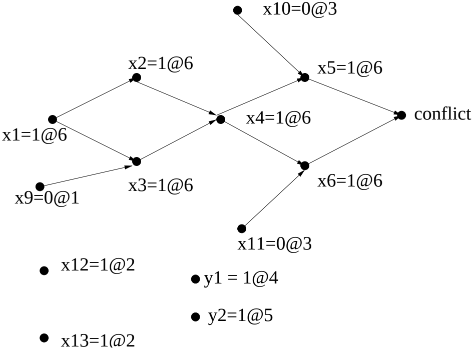

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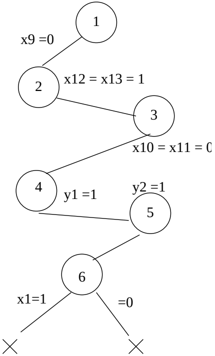

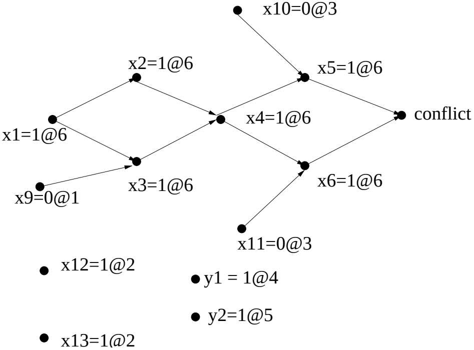

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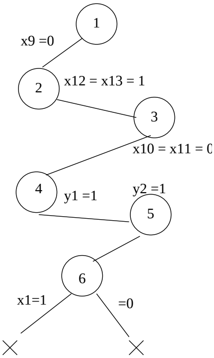
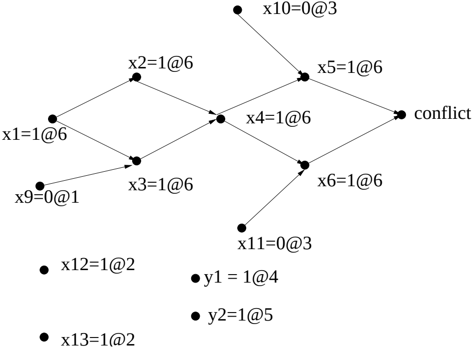
Conflict: $(x_9' \wedge x_{12} \wedge x_{13} \wedge x_{10}' \wedge x_{11}' \wedge y_1 \wedge y_2 \wedge x_1) \implies \text{FALSE}$





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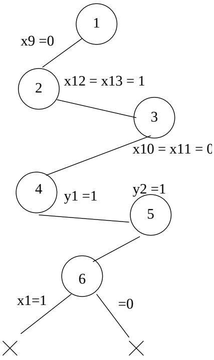
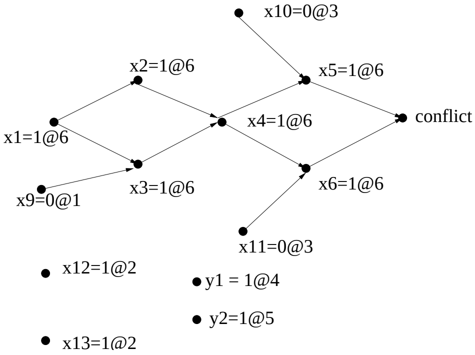
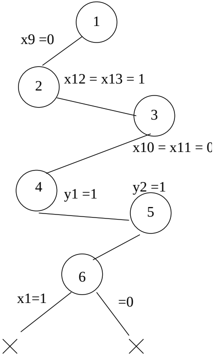
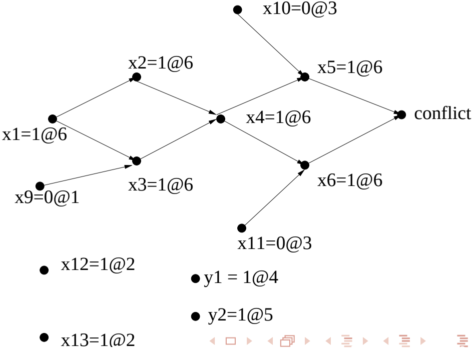
Is the learnt Clause $= (x_9 \vee x_{12}' \vee x_{13}' \vee x_{10} \vee x_{11} \vee y_1' \vee y_2' \vee x_1')$?





- From the conflict-node in the implication graph, traverse back to *antecedents* (or root nodes x_1, x_9, x_{10}, x_{11})
- Note than x_{12}, x_{13}, y_1, y_2 are *unreachable*
- Conflict clause can be simplified:
  - From $(x_9 \vee x_{12}' \vee x_{13}' \vee x_{10} \vee x_{11} \vee y_1' \vee y_2' \vee x_1')$
  - To $(x_9 \vee x_{10} \vee x_{11} \vee x_1')$

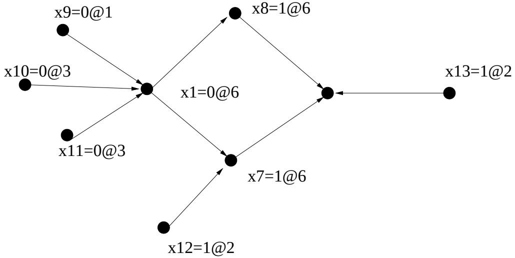




- Add learnt clause to original CNF
- Chronological backtrack: revert last assignment from $x_1 = 1$ to $x_1 = 0$

$$(x_1' + x_2)(x_1' + x_3 + x_9)(x_2' + x_3' + x_4)(x_4' + x_5 + x_{10})(x_4' + x_6 + x_{11})$$
$$(x_5' + x_6')(x_1 + x_7 + x_{12}')(x_1 + x_8)(x_7' + x_8' + x_{13}')(y_1 + z_1)(y_2 + z_2)$$

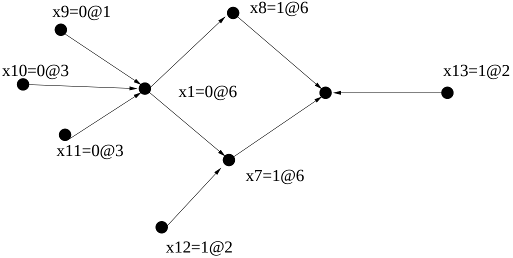
Assignment on Learnt Clause: $(x_9 \vee x_{10} \vee x_{11} \vee x_1')$



- Add learnt clause to original CNF
- Chronological backtrack: revert last assignment from $x_1 = 1$ to $x_1 = 0$

$$(x_1' + x_2)(x_1' + x_3 + x_9)(x_2' + x_3' + x_4)(x_4' + x_5 + x_{10})(x_4' + x_6 + x_{11})$$
$$(x_5' + x_6')(x_1 + x_7 + x_{12}')(x_1 + x_8)(x_7' + x_8' + x_{13}')(y_1 + z_1)(y_2 + z_2)$$

Assignment on Learnt Clause: $(x_9 \vee x_{10} \vee x_{11} \vee x_1')$



$x_1 = 0$ also leads to a conflict. Learn new clause?

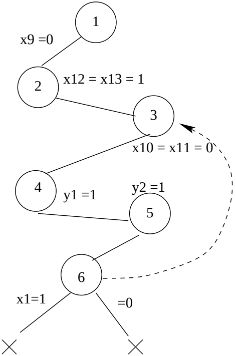
$$(x_1' + x_2)(x_1' + x_3 + x_9)(x_2' + x_3' + x_4)(x_4' + x_5 + x_{10})(x_4' + x_6 + x_{11})$$
$$(x_5' + x_6')(x_1 + x_7 + x_{12}')(x_1 + x_8)(x_7' + x_8' + x_{13}')(y_1 + z_1)(y_2 + z_2)$$

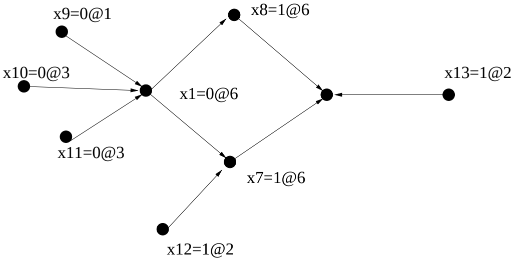
First learnt/conflict clause CC_1: $(x_9 \vee x_{10} \vee x_{11} \vee x_1')$

- New conflict clause also derived from implication graph
- CC_2: $(x_9 \vee x_{12}' \vee x_{13}' \vee x_{10} \vee x_{11})$
- Decision on x_1, y_1, y_2 does not affect the CNF SAT!
- Non-Chronological backtrack:
  - To the MAX decision-level in the conflict clause!
  - Backtrack to Decision-Level 3, undo x_{10} or x_{11}

CDCL search space pruning

CC_2: $(x_9 \vee x'_{12} \vee x'_{13} \vee x_{10} \vee x_{11})$





Conflict-Driven Clause Learning (CDCL) solvers

- Recent techniques can identify more conflict clauses
- Identify unique implication points (UIPs)
- Decision heuristics: Branch on high-activity literals [GRASP]
  - Activity: A score for every literal
  - The number of occurrences of a literal in the formula
- As conflict clauses are added, activity changes
- After n conflicts, multiply activity by $f < 1$, or *rescore*
  - VSIDS heuristic: Variable State Independent Decaying Sum [CHAFF]

A List of CDCL SAT solvers

- GRASP, circa 1996, from Silva and Sakallah
- zCHAFF 2001, from Princeton, Prof. Sharad Malik
- BerkMin 2002
- MiniSAT, 2004 (?) from Cadence Berkeley Labs
- PicoSAT and Lingeling, from Prof. Armin Biere, Univ. Linz
- Please visit `www.satisfiability.org`

Extract UNSAT Cores from UNSAT CNF

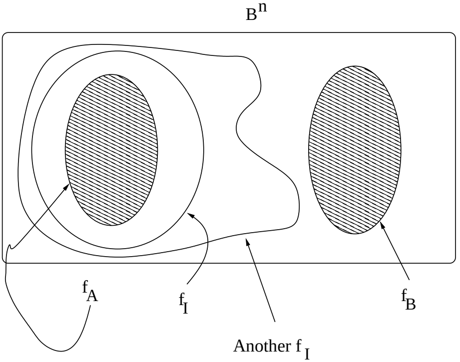
- CNF: $\mathcal{F} = (a' + b')(a' + b)(a + b')(a + b)(x + y)(y + z)$
  - Note that $\mathcal{F}$ is UNSAT
  - Identify a minimum number of clauses that make $\mathcal{F}$ UNSAT
  - This subset of clauses is the *UNSAT Core*, or MIN-UNSAT
  - Helps to identify the causes for UNSAT
- $(a' + b')(a' + b)(a + b')(a + b)$ is the UNSAT core in $\mathcal{F}$
- UNSAT core may not be unique
- UNSAT cores have many applications in verification
- Study of UNSAT cores and applications: Potential class project option!

Craig Interlants: A concept of "abstraction", for UNSAT problems

Definition

Let $f(X_A, X_B, X_C)$ be a Boolean function in variables $X = \{x_1, \dots, x_n\}$ such that X is partitioned into disjoint subsets X_A, X_B, X_C. Let $f = f_A(X_A, X_C) \wedge f_B(X_B, X_C) = \emptyset$. Then there exists another Boolean function f_I such that:

- $f_A \implies f_I$; or $f_A \subseteq f_I$
- $f_I \wedge f_B = \emptyset$
- $f_I(X_C)$ only contains X_C variables, i.e. the common variables of f_A, f_B: $Vars(f_I) \subseteq Vars(f_A) \cap Vars(f_B)$



- The ABC tool with MiniSAT solver can return an f_I, provided f_A, f_B, X_A, X_B, X_C is given.
- Interpolant computed through a resolution proof

Craig Interpolants: Examples

There may be more than one interpolants:

- $f = f_A \cdot f_B$
- $f_A = (a_1 + a_2')(a_1' + a_3')(a_2) = a_1 a_2 a_3'$
- $f_B = (a_2' + a_3)(a_2 + a4)(a_4') = a_2 a_3 a_4'$
- $X_A = \{a_1\}, X_B = \{a_4\}, X_C = \{a_2, a_3\}$
- One interpolant $f_{I_1} = a_3' a_2$
- Another interpolant $f_{I_2} = a_3'$
- The set of all interpolants forms a **lattice**, the smallest interpolant at the bottom, and the largest at the top
- Smallest interpolant: $f_I^{smallest} = \exists_{X_A} f_A(X_A, X_C)$
- Largest interpolant: $f_I^{largest} = \overline{\exists_{X_B} f_B(X_B, X_C)}$

Craig Interpolant: Examples

From the previous slide:

- $f_A = a_1 a_2 a_3'$ and $f_B = a_2 a_3 a_4'$
- $X_A = \{a_1\}, X_B = \{a_4\}, X_C = \{a_2, a_3\}$
- Smallest interpolant: $f_I^{smallest} = \exists_{X_A} f_A(X_A, X_C) = a_2 a_3'$
- Largest interpolant: $f_I^{largest} = \overline{\exists_{X_B} f_B(X_B, X_C)}$
  - Largest interpolant $= \overline{\exists_{X_B} a_2 a_3 a_4'} = \overline{a_2 a_3} = a_2' + a_3'$
- Let f_I be any interpolant, then $f_I^{smallest} \subseteq f_I \subseteq f_I^{largest}$
- Question in the exam: How many interpolants exist for a given (f_A, f_B) pair? Can you find all of them?

- Where does SAT fail?
- For hard UNSAT instances, such as equivalence verification

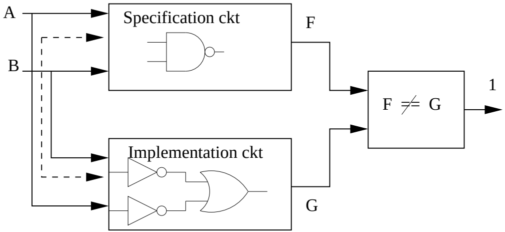


Figure: Miter the circuits F, G

- Prove UNSAT, or find a counter-example
- Limitations: No internal structural equivalences
- EDA-techniques: Circuit-SAT, AIG-reductions, constraint-learning

What Next?

How to improve SAT for Circuit Equivalence Verification?

AND-INVERT-GRAPH (AIG) based Reductions!