$J \subset R$, $J = \langle f_1, \ldots, f_s \rangle$

$R = \mathbb{F}[\underbrace{x_1, \ldots, x_d}_{x}]$

$$\left.\begin{array}{l} f_1(x) = 0 \\ f_2(x) = 0 \\ \vdots \\ f_s(x) = 0 \end{array}\right\}$$

Common zeros = ? ✓

J = ideal generated by $\{f_1, \ldots, f_s\}$

$\longrightarrow$ Gröbner basis $\{g_1, \ldots, g_t\}$

$$\left.\begin{array}{l} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_t(x) = 0 \end{array}\right\}$$

Common zeros? ✓

Common zeros = Variety.

$$V(f_1, \dots, f_s) = V(g_1, \dots, g_t)$$

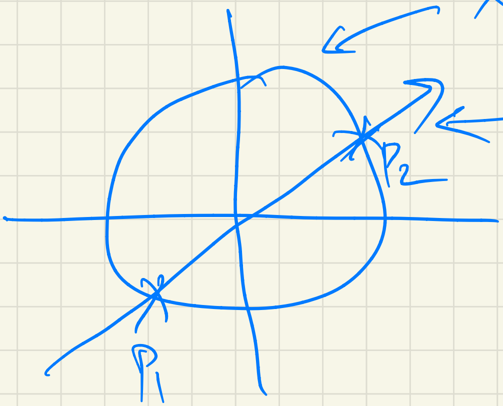
$$= V(J)$$

$f_1 = x^2 + y^2 - 1 = 0$

$f_2 = -x + y = 0$

$V(f_1, f_2) =$

$x^2 + y^2 = 1$



$y - x = 0$

$y^2 = x^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{2}$

$y = \pm\sqrt{2}$

$P_1 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$P_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$V(J) = \{P_1, P_2\}$

variety depends not just on the given set of polynomials $\{f_1 \dots f_s\}$ but on the ideal generated by them: $J = \langle f_1, \dots f_s \rangle$

$$V(f_1, \dots f_s) = V(J)$$

$$\downarrow \text{G.B.}$$

$$V(g_1 \dots g_t) = V(J)$$

$$
\begin{array}{r|l}
 & y \\
\hline
f_1 \;\; yx - y & y^2x - x \\
 & -\left(\cancel{y^2x} - y^2\right) \\
\hline
 & y^2 - x = r_1
\end{array}
$$

$$
\begin{array}{r|l}
 & 1 \\
\hline
f_2 = y^2 - x & y^2 - x \\
 & y^2 - x \\
\hline
 & 0
\end{array}
$$

$$f = y f_1 + 1 \cdot f_2$$

$$f \xrightarrow{f_1} y^2 - x \xrightarrow{f_2} 0$$

$$f_2 = y^2 - x$$

$$\begin{array}{r} x \\ \hline y^2 x - x \enclose{longdiv}{} \end{array}$$

$y^2 x - x$

$x - x^2$

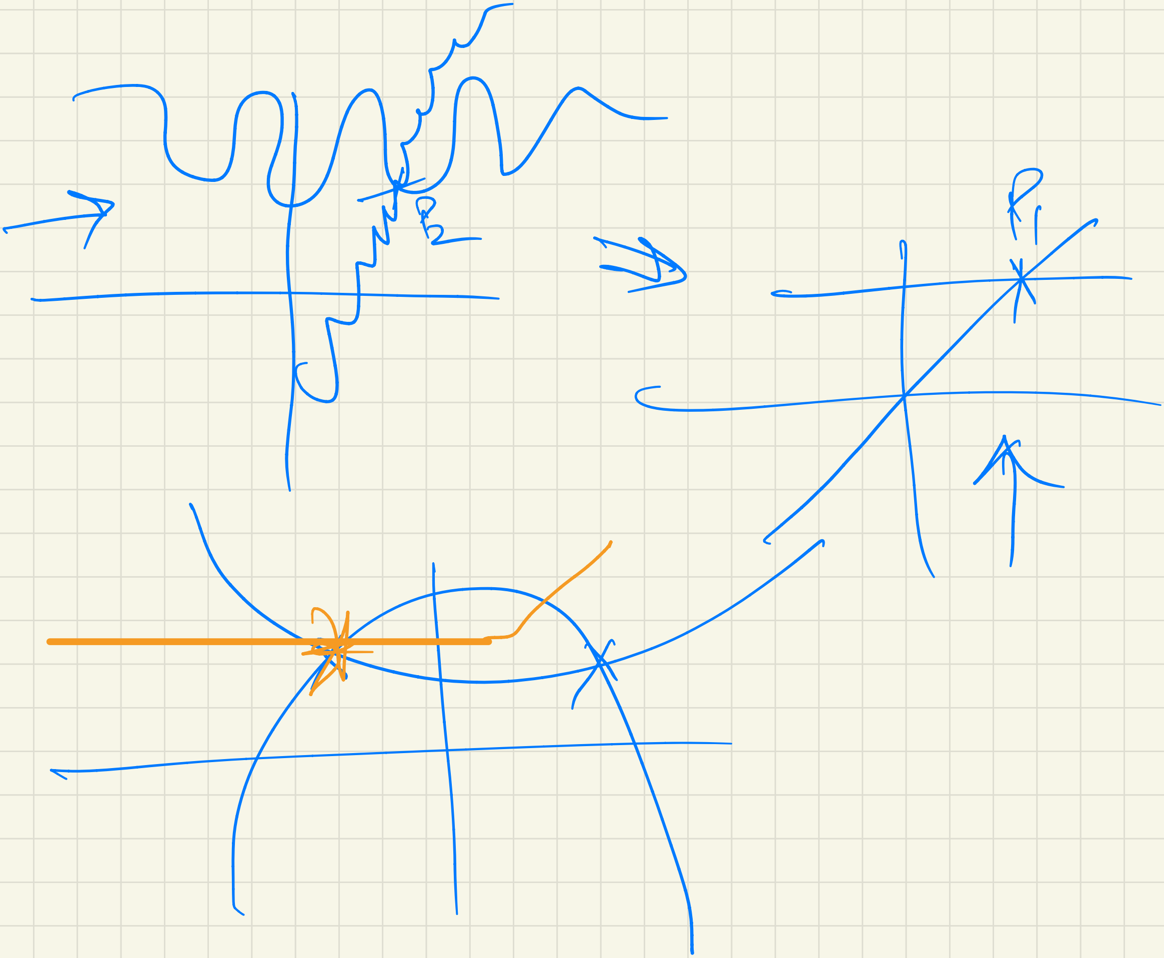
$+$

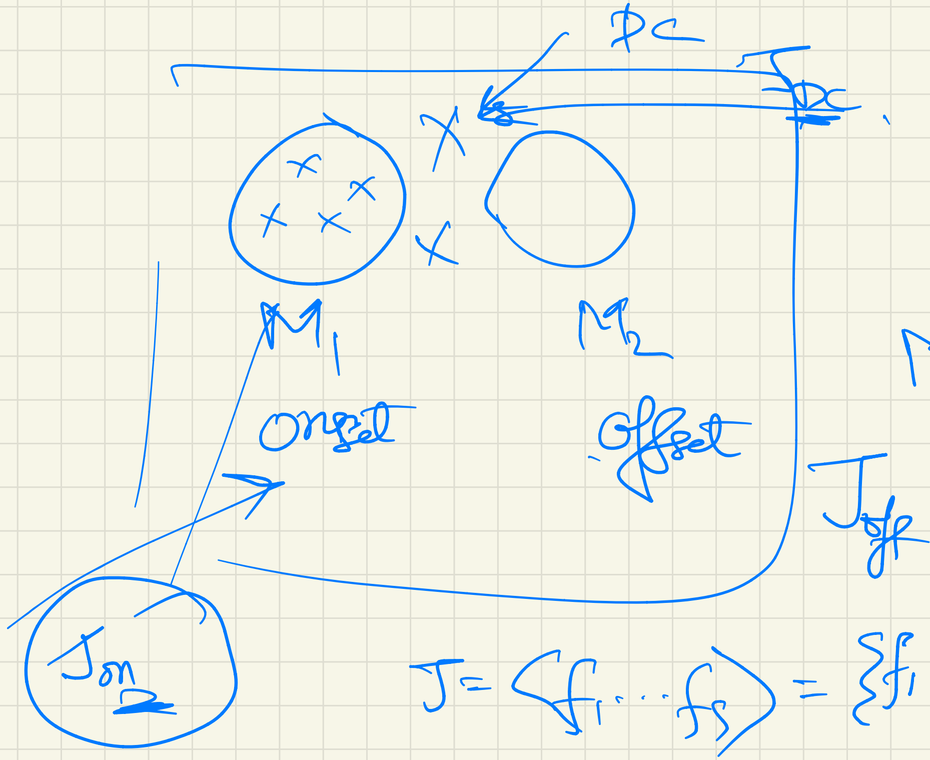
$x^2 - x = r_1$

$f_1 = yx - y \,\Big|\, x^2 - x$

$(yx)\left(\frac{x}{y}\right)$

not polynomial


P2
P1



$M_1 \cap M_2 = \emptyset$

$J = \langle f_1 \cdots f_s \rangle = \{ f_1 \cdot h_1 + \cdots + h_s f_s \}$

$s = 1 \quad f_1 = 1$

$J = \langle f_1 = 1 \rangle = \{ 1 \cdot h_1 \} = F[x_1 \cdots x_d]$