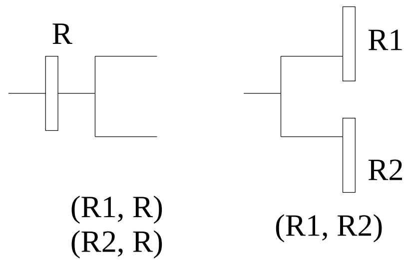
Lecture 2

- Introduction to Sets, Relations, Functions & Graphs
- Boolean logic as Sets and their Graph Representations
- Set = collection of objects; Cardinality of $S = |S|$
- $S = \{a, b, c\}, |S| = 3$
- Notation: $\in, \subset, \supset, \cup, \cap$
- Complement of a set X is $\overline{X}$
- Universe $U = \{1, 2, 3, 4\}, X = \{1, 3\}, \overline{X} = \{2, 4\}$
- $|A \cup B| = |A| + |B| - |A \cap B|$
- Set Difference: $A - B = A \cap \overline{B}$

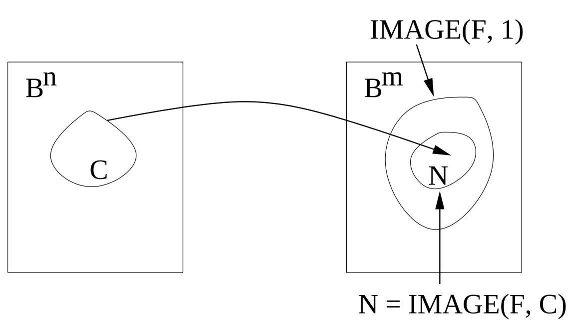
Relations

- Cartesian product $X \times Y$, set of ALL ordered pairs (x, y)
- $X = \{x_1, x_2\}, Y = \{y_1, y_2\}, X \times Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$
- Relation R between sets A and B is a mapping between them
  - $R \subset A \times B$, written aRb, $a \in A, b \in B, (a, b) \in R$.
  - Reflexive: $(a, a) \in R$
  - Symmetric: $(a, b) \in R \Rightarrow (b, a) \in R$
  - Transitive:$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$



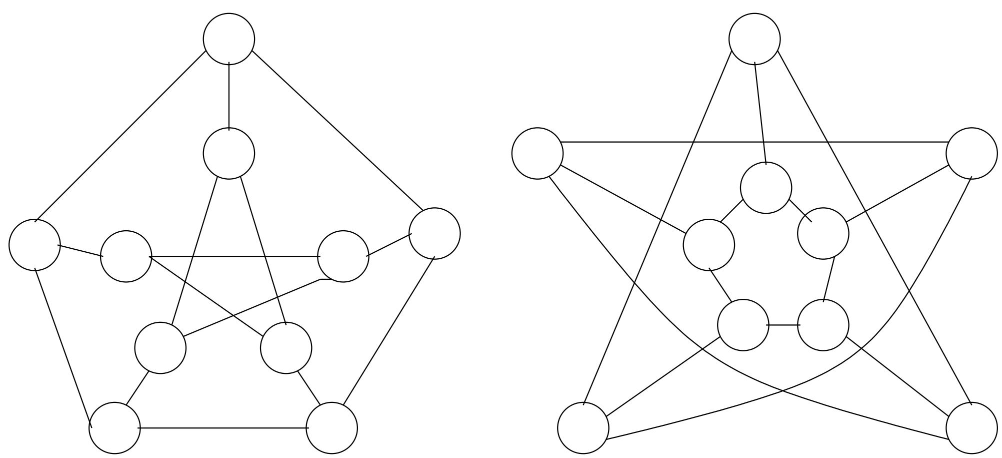
Functions

- Function f: mapping between sets A, B, $f : A \rightarrow B$
- A is the domain of f, B is the co-domain
- $y = f(x), x \in A, y \in B$ then $f(x)$ is codomain or range?
- **Image of a function**
  - $y = f(x) : A \rightarrow B, x \in A, y \in B$
  - y is the image of x under f
  - Let $C \subseteq A$, then $IMG(f, C) = \{f(x) | \forall x \in C\}$



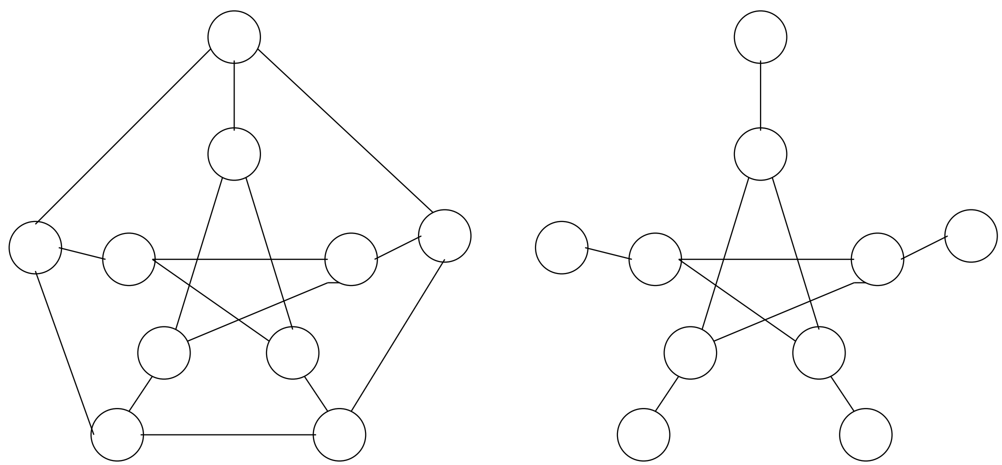
Graph Theory

- Graph $G(V, E)$ is a pair (V, E) where V, E are sets of vertices and edges.
- Edge $e_{ij} = (v_i, v_j) \in R; v_i, v_j \in V; e_{ij} \in E$
- When Edge e meets Vertex v: e is **incident** on v
- **Degree** of v = number of incident edges.



Traversing a Graph

- **Walk**: Sequence of v and e
- **Path**: Walk with distinct vertices (no repetition of v)
- **Connected Graph**: From each vertex, there is a path to all vertices



- Directions on edges == directed graph or **digraph**

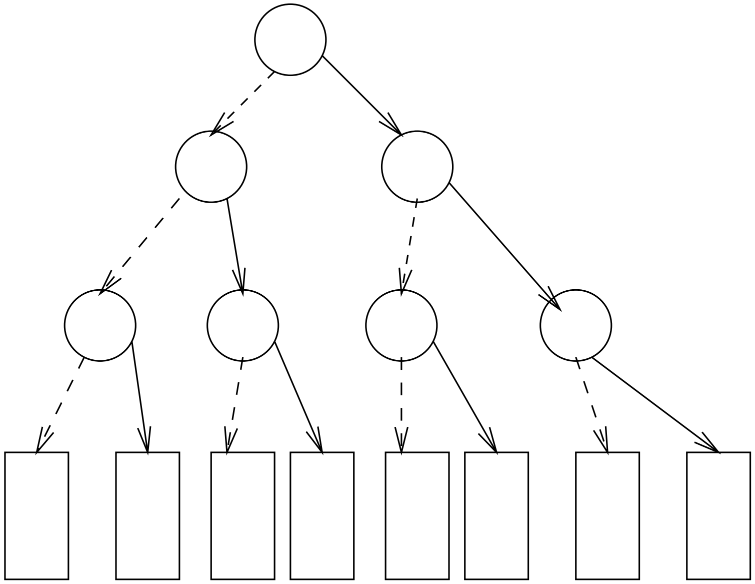
Circuits as Graphs

- Represent gates as vertices (nodes), wires as **directed** edges
- Inputs and outputs are also represented as vertices
- A path from input-to-output has no duplicated vertices (gates)
- Delay of a circuit == **longest path** from input to output
- Directed graph with no cycles = Directed Acyclic Graph!
- What if there are directed cycles in a graph corresponding to a circuit?

Trees as Graphs

- Tree = Connected acyclic graph
- Binary Tree = Tree with only one vertex of degree 2, and all other vertices of degree 3 or 1
- Truth Table versus Binary Tree

a	b	c	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

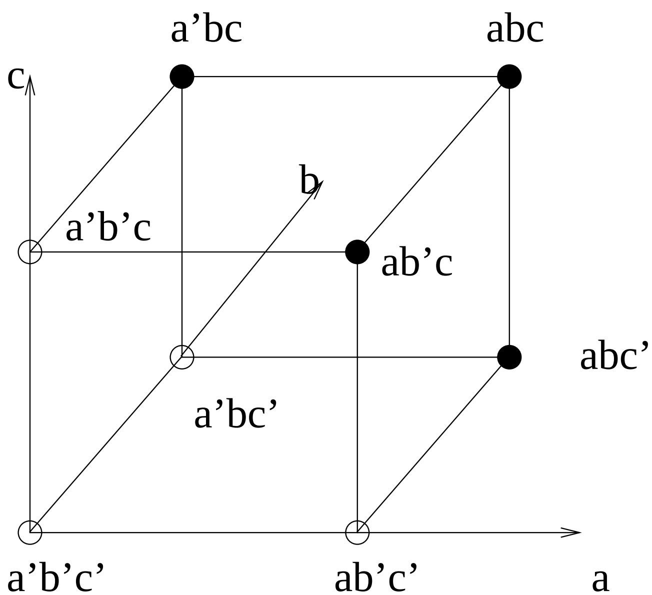


Boolean Algebra - "Basics"

- Algebra is defined over a **set** along with operations on it
- Boolean Algebra: set $= \mathbf{B} \equiv \{0, 1\}$, OPS: $+$ and $\cdot$
- Complement: $a \subset B$, denoted by a' or $\overline{a}$
- Boolean Function is a mapping between domain and co-domain
  - Completely specified Boolean function $f : B^n \rightarrow B$
  - n = number of variables of the system
  - B^n = Boolean space spanned by n variables: n-D **cube**
- Multi-output Boolean $f : B^n \rightarrow B^m$
- Incompletely specified Boolean function: input space is $\subset B^n$

n-Dimensional Cube

$$f = a'bc + ab'c + abc' + abc$$
$$f = ab + bc + ac$$



bc / a

		1	
	1	1	1

Terminology

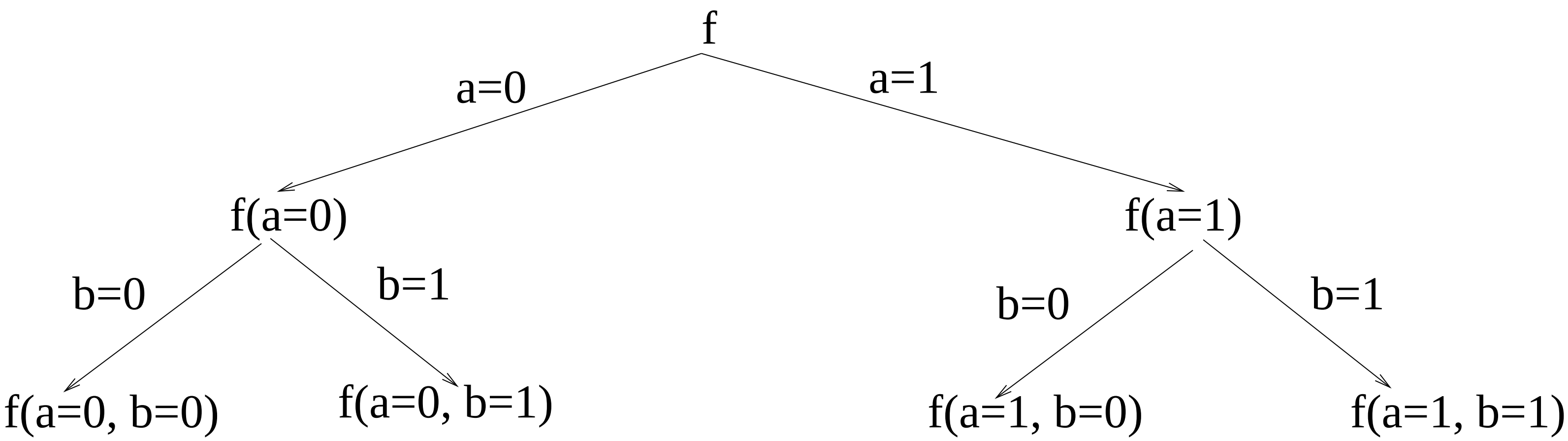
- **Binary Variable**= symbol. Represents a co-ordinate of B^n
- **Literal**: Boolean variable OR its complement
- $f = a + a'b$ has how many literals?
- Objective of Logic Optimization: Minimization of literals
- **Cube**: a point, or a set of points in B^n
- Terminology comes from n-D cube
- $a'bc + abc$: 2 cubes. bc is a smaller or larger cube?
- **Support** variables of a function....

Co-Factors of a Boolean Function

- $f(x_1, \ldots x_i, \ldots x_n)$ be a function
- $f_{x_i} = f(x_1, \ldots x_i = 1, \ldots, x_n)$ is positive cofactor of f
- $f_{\overline{x_i}} = f(x_1, \ldots x_i = 0, \ldots, x_n)$ is negative cofactor of f
- $f = ac + bc + ab$
- $f_a = f_{a=1} =$
- $f_{\overline{a}} = f_{a=0} =$
- Co-factor of f has more or few or equal number of literals?
- Cofactors of f w.r.t. x DO NOT contain x
- Shannon's (Boole's) Theorem:
  - $f = x \cdot f_x + x' \cdot f_{x'}$

Further Co-factoring...

- $f_a = f_{a=1} = b + c$
- $f_{\overline{a}} = f_{a=0} = bc$
- $f = af_a + a'f_{a'} = a(b+c) + a'(bc)$



Monotonicity of Boolean Functions

- $f(x_1, \ldots, x_i, \ldots, x_n)$ is positive **unate in variable** x_i if $f_{x_i} \supseteq f_{\overline{x}_i}$
- $f(x_1, \ldots, x_i, \ldots, x_n)$ is negative **unate in variable** x_i if $f_{x_i} \subseteq f_{\overline{x}_i}$
- $f = a + b + c'$
- $f_a = 1 + b + c = 1 \equiv$
- $f_{a'} = b + c' =\equiv$
- The **minterms** of f_a **contain the minterms of** $f_{a'}$
- If $f_a \supseteq f_{a'}$ then
  - $f = af_a + f_{a'}$
  - What if $f_a \subseteq f_{a'}$?