An Introduction to Probability Theory and Its Applications

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tical lines $x = \frac{3}{2}$ and $x = n - \frac{1}{2}$. Now $\frac{1}{2}\log n$ quite obviously exceeds the area of the strip $n - \frac{1}{2} < x < n$ under the curve, and hence a_n *exceeds* the area under the curve and between $x = \frac{3}{2}$ and $x = n$. In other words, we have shown that

$$(9.4) \qquad \int_{\frac{3}{2}}^{n} \log x \cdot dx < a_n < \int_{1}^{n} \log x \cdot dx.$$

The indefinite integral of $\log x$ is given by $x \log x - x$, and equation (9.4) reduces to the double inequality

$$(9.5) \quad (n + \tfrac{1}{2}) \log n - n + \tfrac{3}{2}(1 - \log \tfrac{3}{2}) < \\ < \log n! < (n + \tfrac{1}{2}) \log n - n + 1.$$

Put for abbreviation

$$(9.6) \qquad \delta_n = \log n! - (n + \tfrac{1}{2}) \log n + n.$$

Then $1 - \delta_n$ is the difference between the extreme right member of (9.5) and $\log n!$, that is, $1 - \delta_n$ equals the area of the domain between the curve $y = \log x$ and the polygon $A_1 A_2 \ldots A_n$. It follows that δ_n *decreases monotonically*. But by (9.5) we have $\frac{3}{2}(1 - \log \frac{3}{2}) < \delta_n < 1$. We conclude that δ_n tends to a limit comprised between 1 and $\frac{3}{2}(1 - \log \frac{3}{2})$. Denoting this limit by $\log c$ we have

$$(9.7) \qquad \delta_n \to \log c \qquad \text{where} \quad 2.45 < c < 2.72.$$

In logarithmic notation Stirling's formula reduces to (9.7) with $c = (2\pi)^{\frac{1}{2}}$ (or 2.507, approximately). Now π can be defined in many ways, and for our purposes it is simplest and most natural to *define* $\pi = c^2/2$. With this definition we have Stirling's formula, but it remains to show that the constant so defined agrees with the more familiar π of other formulas. This fact will develop as a by-product of other calculations in chapter VII, and so the proof of Stirling's formula will be completed there.

Refinements. Stirling's formula can be improved by the addition of further terms. Although we shall never make use of such refinements, we shall here indicate the proof of the following *double inequality* [15]

$$(9.8) \qquad (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n+1/(12n+1)} < n! < (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n+1/(12n)}.$$

To prove (9.8) note that

$$(9.9) \quad \delta_n - \delta_{n+1} = \left(n + \frac{1}{2}\right) \log \frac{n+1}{n} - 1 = \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^5} + \ldots$$

[15] H. Robbins, A remark on Stirling's formula, *American Mathematical Monthly*, vol. 62 (1955), pp. 26–29.

[the last expansion follows from (8.11) on setting $t = 1/(2n+1)$]. We *increase* the extreme right member in (9.9) by replacing the coefficients $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$ by $\frac{1}{3}$; this leads to a geometric series with ratio $(2n+1)^{-2}$, and thus

$$(9.10) \qquad \delta_n - \delta_{n+1} < \frac{1}{3[(2n+1)^2 - 1]} = \frac{1}{12n} - \frac{1}{12(n+1)}.$$

Accordingly, $\delta_n - 1/12n$ *increases* monotonically. Now the limit of this sequence is given by Stirling's formula, and passing to antilogarithms we have the second inequality in (9.8). The first inequality follows similarly from (9.9) on noticing that

$$(9.11) \qquad \delta_n - \delta_{n+1} > \frac{1}{3(2n+1)^2} > \frac{1}{12n+1} - \frac{1}{12(n+1)+1}.$$

The accuracy of the approximations (9.8) is remarkable; even for $n = 1$ the formula leads to the two bounds 0.9958... and 1.0023.... The upper bound provided in (9.8) is slightly better [cf. (12.28)]. For $n = 2$ it yields 2.0007, for $n = 5$ we get 120.01..., and for $n = 10$ the first five significant figures are correct.

PROBLEMS FOR SOLUTION

Note: Sections 11 and 12 contain problems of a different character and diverse complements to the text.

10. EXERCISES AND EXAMPLES

Note: *Assume in each case that all arrangements have the same probability.*

1. How many different sets of initials can be formed if every person has one surname and (*a*) exactly two given names, (*b*) at most two given names, (*c*) at most three given names?

2. In how many ways can two rooks of different colors be put on a chessboard so that they can take each other?

3. Letters in the Morse code are formed by a succession of dashes and dots with repetitions permitted. How many letters is it possible to form with ten symbols or less?

4. Each domino piece is marked by two numbers. The pieces are symmetrical so that the number-pair is not ordered. How many different pieces can be made using the numbers $1, 2, \ldots, n$?

5. The numbers $1, 2, \ldots, n$ are arranged in random order. Find the probability that the digits (*a*) 1 and 2, (*b*) 1, 2, and 3, appear as neighbors in the order named.

6. (*a*) Find the probability that among three random digits there occur 2, 1, or 0 repetitions. (*b*) Do the same for four random digits.

7. Find the probabilities p_r that in a sample of r random digits no two are equal. Estimate the numerical value of p_{10}, using Stirling's formula.

8. What is the probability that among k random digits (*a*) 0 does not appear; (*b*) 1 does not appear; (*c*) neither 0 nor 1 appears; (*d*) at least one of the two digits 0 and 1 does not appear? Let A and B represent the events in (*a*) and (*b*). Express the other events in terms of A and B.

9. If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

10. At a parking lot there are twelve places arranged in a row. A man observed that there were eight cars parked, and that the four empty places were adjacent to each other (formed *one* run). Given that there are four empty places, is this arrangement surprising (indicative of non-randomness)?

11. A man is given n keys of which only one fits his door. He tries them successively (sampling without replacement). This procedure may require 1, 2, ..., n trials. Show that each of these n outcomes has probability n^{-1}.

12. Suppose that each of n sticks is broken into one long and one short part. The $2n$ parts are arranged into n pairs from which new sticks are formed. Find the probability (a) that the parts will be joined in the original order, (b) that all long parts are paired with short parts.[16]

13. *Testing a statistical hypothesis.* A Cornell professor got a ticket twelve times for illegal overnight parking. All twelve tickets were given either Tuesdays or Thursdays. Find the probability of this event. (Was his renting a garage only for Tuesdays and Thursdays justified?)

14. *Continuation.* Of twelve police tickets none was given on Sunday. Is this evidence that no tickets are given on Sundays?

15. A box contains ninety good and ten defective screws. If ten screws are used, what is the probability that none is defective?

16. From the population of five symbols a, b, c, d, e, a sample of size 25 is taken. Find the probability that the sample will contain five symbols of each kind. Check the result in tables of random numbers,[17] identifying the digits 0 and 1 with a, the digits 2 and 3 with b, etc.

17. If n men, among whom are A and B, stand in a row, what is the probability that there will be exactly r men between A and B? If they stand in a ring instead of in a row, show that the probability is independent of r and hence $1/(n-1)$. (In the circular arrangement consider only the arc leading from A to B in the positive direction.)

18. What is the probability that two throws with three dice each will show the same configuration if (a) the dice are distinguishable, (b) they are not?

19. Show that it is more probable to get at least one ace with four dice than at least one double ace in 24 throws of two dice. (The answer is known as de Méré's paradox. Chevalier de Méré, a gambler, thought that the two probabilities ought to be equal and blamed mathematics for his losses.)

20. From a population of n elements a sample of size r is taken. Find the probability that none of N prescribed elements will be included in the sample,

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[16] When cells are exposed to harmful radiation, some chromosomes break and play the role of our "sticks." The "long" side is the one containing the so-called centromere. If two "long" or two "short" parts unite, the cell dies. See D. G. Catcheside, The effect of X-ray dosage upon the frequency of induced structural changes in the chromosomes of *Drosophila Melanogaster*, *Journal of Genetics*, vol. 36 (1938), pp. 307–320.

[17] They are occasionally extraordinarily obliging: see J. A. Greenwood and E. E. Stuart, Review of Dr. Feller's critique, *Journal for Parapsychology*, vol. 4 (1940), pp. 298–319, in particular p. 306.

assuming the sampling to be (*a*) without, (*b*) with replacement. Compare the numerical values for the two methods when (i) $n = 100$, $r = N = 3$, and (ii) $n = 100$, $r = N = 10$.

21. *Spread of rumors.* In a town of $n + 1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, etc. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor will be told r times without: (*a*) returning to the originator, (*b*) being repeated to any person. Do the same problem when at each step the rumor is told by one person to a gathering of N randomly chosen people. (The first question is the special case $N = 1$.)

22. *Chain letters.* In a population of $n + 1$ people a man, the "progenitor," sends out letters to two persons, the "first generation." These repeat the performance and, generally, each member of the rth generation sends out letters to two persons chosen at random. Find the probability that the generations number 1, 2, ..., r will not include the progenitor. Find the median of the distribution, supposing n to be large.

23. *A familiar problem.* In a certain family four girls take turns at washing dishes. Out of a total of four breakages, three were caused by the youngest girl, and she was thereafter called clumsy. Was she justified in attributing the frequency of her breakages to chance? Discuss the connection with random placements of balls.

24. What is the probability that (*a*) the birthdays of twelve people will fall in twelve different calendar months (assume equal probabilities for the twelve months), (*b*) the birthdays of six people will fall in exactly two calendar months?

25. Given thirty people, find the probability that among the twelve months there are six containing two birthdays and six containing three.

26. A closet contains n pairs of shoes. If $2r$ shoes are chosen at random (with $2r < n$), what is the probability that there will be (*a*) no complete pair, (*b*) exactly one complete pair, (*c*) exactly two complete pairs among them?

27. A car is parked among N cars in a row, not at either end. On his return the owner finds that exactly r of the N places are still occupied. What is the probability that both neighboring places are empty?

28. A group of $2N$ boys and $2N$ girls is divided into two equal groups. Find the probability p that each group will be equally divided into boys and girls. Estimate p, using Stirling's formula.

29. In bridge, prove that the probability p of West's receiving exactly k aces is the same as the probability that an arbitrary hand of thirteen cards contains exactly k aces. (This is intuitively clear. Note, however, that the two probabilities refer to two different experiments, since in the second case thirteen cards are chosen at random and in the first case all 52 are distributed.)

30. The probability that in a bridge game East receives m and South n spades is the same as the probability that of two hands of thirteen cards each, drawn at random from a deck of bridge cards, the first contains m and the second n spades.

31. What is the probability that the bridge hands of North and South together contain exactly k aces, where $k = 0, 1, 2, 3, 4$?

32. Let a, b, c, d be four non-negative integers such that $a + b + c + d = 13$. Find the probability $p(a, b, c, d)$ that in a bridge game the players North, East,

South, West have a, b, c, d spades, respectively. Formulate a scheme of placing red and black balls into cells that contains the problem as a special case.

33. Using the result of problem 32, find the probability that some player receives a, another b, a third c, and the last d spades if (a) $a = 5$, $b = 4$, $c = 3$, $d = 1$; (b) $a = b = c = 4$, $d = 1$; (c) $a = b = 4$, $c = 3$, $d = 2$.

Note that the three cases are essentially different.

34. Let a, b, c, d be integers with $a + b + c + d = 13$. Find the probability $q(a, b, c, d)$ that a hand at bridge will consist of a spades, b hearts, c diamonds, and d clubs and show that the problem does *not* reduce to one of placing, at random, thirteen balls into four cells. Why?

35. *Distribution of aces among r bridge cards.* Calculate the probabilities $p_0(r)$, $p_1(r)$, $\ldots$, $p_4(r)$ that among r bridge cards drawn at random there are 0, 1, $\ldots$, 4 aces, respectively. Verify that $p_0(r) = p_4(52 - r)$.

36. *Continuation: waiting times.* If the cards are drawn one by one, find the probabilities $f_1(r)$, $\ldots$, $f_4(r)$ that the first, $\ldots$, fourth ace turns up at the rth trial. *Guess* at the *medians* of the waiting times for the first, $\ldots$, fourth ace and then calculate them.

37. Find the probability that each of two hands contains exactly k aces if the two hands are composed of r bridge cards each, and are drawn (a) from the same deck, (b) from two decks. Show that when $r = 13$ the probability in part (a) is the probability that two preassigned bridge players receive exactly k aces each.

38. *Misprints.* Each page of a book contains N symbols, possibly misprints. The book contains $n = 500$ pages and $r = 50$ misprints. Show that (a) the probability that pages number 1, 2, $\ldots$, n contain, respectively, r_1, r_2, $\ldots$, r_n misprints equals

$$\binom{N}{r_1}\binom{N}{r_2}\cdots\binom{N}{r_n} \div \binom{nN}{r};$$

(b) for large N this probability may be approximated by (5.5). Conclude that *the r misprints are distributed in the n pages approximately in accordance with a random distribution of r balls in n cells.* (*Note.* This may be restated as a general limiting property of Fermi-Dirac statistics. Cf. section 5.)

Note: *The following problems refer to the material of section 5.*

39. If r_1 indistinguishable things of one kind and r_2 indistinguishable things of a second kind are placed into n cells, find the number of distinguishable arrangements.

40. If r_1 dice and r_2 coins are thrown, how many results can be distinguished?

41. In how many different distinguishable ways can r_1 white, r_2 black, and r_3 red balls be arranged?

42. Find the probability that in a random arrangement of 52 bridge cards no two aces are adjacent.

43. *Elevator.* In the example (3.c) the elevator starts with seven passengers and stops at ten floors. The various arrangements of discharge may be denoted by symbols like (3, 2, 2), to be interpreted as the event that three passengers leave together at a certain floor, two other passengers at another

floor, and the last two at still another floor. Find the probabilities of the fifteen possible arrangements ranging from (7) to (1, 1, 1, 1, 1, 1, 1).

44. *Birthdays.* Find the probabilities for the various configurations of the birthdays of 22 people.

45. Find the probability for a *poker* hand to be a (*a*) royal flush (ten, jack, queen, king, ace in a single suit); (*b*) four of a kind (four cards of equal face values); (*c*) full house (one pair and one triple of cards with equal face values); (*d*) straight (five cards in sequence regardless of suit); (*e*) three of a kind (three equal face values plus two extra cards); (*f*) two pairs (two pairs of equal face values plus one other card); (*g*) one pair (one pair of equal face values plus three different cards).

11. PROBLEMS AND COMPLEMENTS OF A THEORETICAL CHARACTER

1. A population of n elements includes np red ones and nq black ones ($p + q = 1$). A random sample of size r is taken with replacement. Show that the probability of its including exactly k red elements is

$$\binom{r}{k} p^k q^{r-k}. \tag{11.1}$$

2. *A limit theorem for the hypergeometric distribution.* If n is large and $n_1/n = p$, then the probability q_k given by (6.1) and (6.2) is close to (11.1). More precisely,

$$\binom{r}{k}\left(p - \frac{k}{n}\right)^k\left(q - \frac{r-k}{n}\right)^{r-k} < q_k < \binom{r}{k} p^k q^{r-k}\left(1 - \frac{r}{n}\right)^{-r} \tag{11.2}$$

A comparison of this and the preceding problem shows: *For large populations there is practically no difference between sampling with or without replacement.*

3. A random sample of size r *without replacement* is taken from a population of n elements. The probability u_r that N given elements will all be included in the sample is

$$u_r = \binom{n-N}{r-N} \div \binom{n}{r}. \tag{11.3}$$

(The corresponding formula for sampling *with replacement* is given by (11.10) and cannot be derived by a direct argument. For an alternative form of (11.3) cf. problem IV, 9.)

4. *Limiting form.* If $n \to \infty$ and $r \to \infty$ so that $r/n \to p$, then $u_r \to p^N$ (cf. problem 13).

Note: *Problems* 5–13 *refer to the classical occupancy problem* (*Maxwell-Boltzmann statistics*)*: That is, r balls are distributed among n cells and each of the* n^r *possible distributions has probability* n^{-r}.[18]

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[18] Problems 5–19 play a role in quantum statistics, the theory of photographic plates, G-M counters, etc. The formulas are therefore frequently discussed and discovered in the physical literature, usually without a realization of their classical and essentially elementary character. Probably all the problems occur (although in modified form) in the book by Whitworth quoted at the opening of this chapter.

5. The probability p_k that a given cell contains exactly k balls is given by the binomial distribution (4.5). The most probable number is the integer ν such that $(r - n + 1)/n < \nu \leq (r + 1)/n$. (In other words, it is asserted that $p_0 < p_1 < \ldots < p_{\nu-1} \leq p_\nu > p_{\nu+1} > \ldots > p_r$; cf. problem 15.)

6. *Limiting form.* If $n \to \infty$ and $r \to \infty$ so that the average number $\lambda = r/n$ of balls per cell remains constant, then

$$p_k \to e^{-\lambda}\lambda^k/k!. \tag{11.4}$$

This is the *Poisson distribution,* discussed in chapter VI; see problem 16.

7. Let $A(r, n)$ be the number of distributions leaving *none of the n cells empty.* Show by a combinatorial argument that

$$A(r, n+1) = \sum_{k=1}^{r} \binom{r}{k} A(r-k, n). \tag{11.5}$$

Conclude that

$$A(r, n) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (n - \nu)^r. \tag{11.6}$$

Hint: Use induction; assume (11.6) to hold and express $A(r-k, n)$ in (11.5) accordingly. Change the order of summation and use the binomial formula to express $A(r, n+1)$ as the difference of two simple sums. Replace in the second sum $\nu + 1$ by a new index of summation and use (8.6).

Note: *Formula* (11.6) *provides a theoretical solution to an old problem but obviously it would be a thankless task to use it for the calculation of the probability x, say, that in a village of r =* 1900 *people every day of the year is a birthday. In chapter IV, section* 2, *we shall derive* (11.6) *by another method and obtain a simple approximation formula (showing, e.g., that x =* 0.135, *approximately).*

8. Show that *the number of distributions leaving exactly m cells empty is*

$$E_m(r, n) = \binom{n}{m} A(r, n-m) = \binom{n}{m} \sum_{\nu=0}^{n-m} (-1)^\nu \binom{n-m}{\nu} (n - m - \nu)^r. \tag{11.7}$$

9. Show without using the preceding results that *the probability*

$$p_m(r, n) = n^{-r} E_m(r, n)$$

of finding exactly m cells empty satisfies

$$p_m(r+1, n) = p_m(r, n)\frac{n - m}{n} + p_{m-1}(r, n)\frac{m - 1}{n}. \tag{11.8}$$

10. Using the results of problems 7 and 8, show by direct calculation that (11.8) holds. Show that this method provides *a new derivation* (*by induction on r*) of (11.6).

11. From (11.6) and problem 8 conclude that *the probability of finding m or more cells empty is*

$$\binom{n}{m} \sum_{\nu=0}^{n-m} (-1)^\nu \binom{n-m}{\nu} \left(1 - \frac{m+\nu}{n}\right)^r \frac{m}{m+\nu}. \tag{11.9}$$

(For $m \geqq n$ this expression reduces to zero, as is proper.)

12. *The probability that each of N given cells is occupied is*

$$(11.10)\qquad u(r, n) = n^{-r} \sum_{k=0}^{r} \binom{r}{k} A(k, N)(n-N)^{r-k}$$

Conclude that

$$(11.11)\qquad u(r, n) = \sum_{\nu=0}^{N} (-1)^{\nu} \binom{N}{\nu} \left(1 - \frac{\nu}{n}\right)^{r}.$$

(Use the binomial theorem. For $N = n$ we have $u(r, n) = n^{-r} A(r, n)$. Note that (11.11) is the analogue of (11.3) for *sampling with replacement.*[19] For an alternative derivation see problem IV, 8.)

13. *Limiting form.* For the passage to the limit described in problem 4 one has $u(r, n) \to (1 - e^{-p})^N$.

Note: *In problems 14–19 r and n have the same meaning as above, but we assume that the balls are indistinguishable and that all distinguishable arrangements have equal probabilities (Bose-Einstein statistics).*

14. The probability that a given cell contains exactly k balls is

$$(11.12)\qquad q_k = \binom{n+r-k-2}{r-k} \div \binom{n+r-1}{r}.$$

15. Show that when $n > 2$ zero is the most probable number of balls in any specified cell, or more precisely, $q_0 > q_1 > \ldots$ (cf. problem 5).

16. *Limit theorem.* Let $n \to \infty$ and $r \to \infty$, so that the average number of particles per cell, r/n, tends to λ. Then

$$(11.13)\qquad q_k \to \frac{\lambda^k}{(1+\lambda)^{k+1}}.$$

(The right side is known as the *geometric distribution.*)

17. The probability that exactly m cells remain empty is

$$(11.14)\qquad \rho_m = \binom{n}{m} \binom{r-1}{n-m-1} \div \binom{n+r-1}{r}.$$

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[19] Note that $u(r, n)$ may be interpreted as the probability that the *waiting time* up to the moment when the Nth element joins the sample is less than r. The result may be applied to *random sampling digits:* here $u(r, 10) - u(r-1, 10)$ is the probability that a sequence of r elements must be observed to include the complete set of all ten digits. This can be used as a test of randomness. R. E. Greenwood (Coupon collector's test for random digits, *Mathematical Tables and Other Aids to Computation*, vol. 9 (1955), pp. 1–5) has tabulated the distribution and compared it to actual counts for the corresponding waiting times for the first 2035 decimals of π and the first 2486 decimals of e. The median of the waiting time for a complete set of all ten digits is 27. The probability that this waiting time exceeds 50 is greater than 0.05, and the probability of the waiting time exceeding 75 is about 0.0037.

18. The probability that a group of m prescribed cells contains a total of exactly j balls is

$$q_j(m) = \binom{m+j-1}{m-1}\binom{n-m+r-j-1}{r-j} \div \binom{n+r-1}{r}. \tag{11.15}$$

19. *Limiting form.* For the passage to the limit of problem 4 we have

$$q_j(m) \to \binom{m+j-1}{m-1}\frac{p^j}{(1+p)^{m+j}}. \tag{11.16}$$

(The right side is a special case of the *negative binomial distribution* to be introduced in chapter VI.)

Theorems on Runs. *In problems 20–25 we consider arrangements of r_1 alphas and r_2 betas and assume that all arrangements are equally probable* [*see example* (4.*d*)]. *This group of problems refers to section* 5*a*.

20. The probability that the arrangement contains exactly k runs of either kind is

$$P_{2\nu} = 2\binom{r_1-1}{\nu-1}\binom{r_2-1}{\nu-1} \div \binom{r_1+r_2}{r_1} \tag{11.17}$$

when $k = 2\nu$ is even, and

$$P_{2\nu+1} = \left\{\binom{r_1-1}{\nu}\binom{r_2-1}{\nu-1} + \binom{r_1-1}{\nu-1}\binom{r_2-1}{\nu}\right\} \div \binom{r_1+r_2}{r_1} \tag{11.18}$$

when $k = 2\nu + 1$ is odd.

21. *Continuation.* Conclude that the most probable number of runs is an integer k such that $\frac{2r_1r_2}{r_1+r_2} < k < \frac{2r_1r_2}{r_1+r_2} + 3$. (*Hint*: Consider the ratios $P_{2\nu+2} \div P_{2\nu}$ and $P_{2\nu+1} \div P_{2\nu-1}$.)

22. The probability that the arrangement starts with an alpha run of length $\nu \geq 0$ is $(r_1)_\nu r_2 \div (r_1+r_2)_{\nu+1}$. (*Hint*: Choose the ν alphas and the beta which must follow it.) What does the theorem imply for $\nu = 0$?

23. The probability of having exactly k runs of alphas is

$$\pi_k = \binom{r_1-1}{k-1}\binom{r_2+1}{k} \div \binom{r_1+r_2}{r_1}. \tag{11.19}$$

Hint: This follows easily from the second part of the lemma of section 5. Alternatively, equation (11.19) may be derived from (11.17) and (11.18), but this procedure is more laborious.

24. The probability that the nth alpha is preceded by exactly m betas is

$$\binom{r_1+r_2-n-m}{r_2-m}\binom{m+n-1}{m} \div \binom{r_1+r_2}{r_1}. \tag{11.20}$$

25. The probability for the alphas to be arranged in k runs of which k_1 are of length 1, k_2 of length 2, ..., k_ν of length ν (with $k_1 + \ldots + k_\nu = k$) is

$$\frac{k!}{k_1!k_2!\ldots k_\nu!}\binom{r_2+1}{k} \div \binom{r_1+r_2}{r_1}. \tag{11.21}$$

12. PROBLEMS AND IDENTITIES INVOLVING BINOMIAL COEFFICIENTS

1. For integral $n \geq 2$

$$1 - \binom{n}{1} + \binom{n}{2} - + \ldots = 0$$

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots = n2^{n-1}$$

(12.1)
$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - + \ldots = 0,$$

$$2\cdot 1\binom{n}{2} + 3\cdot 2\binom{n}{3} + 4\cdot 3\binom{n}{4} + \ldots = n(n-1)2^{n-2}$$

(*Hint:* Use the binomial formula.)

2. Prove that for positive integers n, k

(12.2)
$$\binom{n}{0}\binom{n}{k} - \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} \cdots \pm \binom{n}{k}\binom{n-k}{0} = 0.$$

More generally [20]

(12.3)
$$\sum \binom{n}{\nu}\binom{n-\nu}{k-\nu} t^\nu = \binom{n}{k}(1+t)^k.$$

3. For any $a > 0$

(12.4)
$$\binom{-a}{k} = (-1)^k \binom{a+k-1}{k}.$$

If a is an integer, this can be proved also by differentiation of the geometric series $\Sigma x^k = (1-x)^{-1}$.

4. Prove that

(12.5)
$$\binom{2n}{n} 2^{-2n} = (-1)^n \binom{-\frac{1}{2}}{n}.$$

5. For integral non-negative n and r and all real a

(12.6)
$$\sum_{\nu=0}^{n} \binom{a-\nu}{r} = \binom{a+1}{r+1} - \binom{a-n}{r+1}.$$

(*Hint:* Use equation (8.6). The special case $n = a$ is frequently used.)

6. For arbitrary a and integral $n \geq 0$

(12.7)
$$\sum_{\nu=0}^{n} (-1)^\nu \binom{a}{\nu} = (-1)^n \binom{a-1}{n}.$$

[*Hint:* Use equation (8.6).]

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[20] The reader is reminded of the convention (8.5): if ν runs through *all* integers, only finitely many terms in the sum in (12.3) are different from zero.

7. For positive integers r, k

$$\sum_{\nu=0}^{r} \binom{\nu + k - 1}{k - 1} = \binom{r + k}{k}. \tag{12.8}$$

(*a*) Prove this using (8.6). (*b*) Show that (12.8) is a special case of (12.7). (*c*) Show by an inductive argument that (12.8) leads to a new proof of the first part of the lemma of section 5.

8. In section 6 we remarked that the terms of the hypergeometric distribution should add to unity. This amounts to saying that for any positive integers a, b, n,

$$\binom{a}{0}\binom{b}{n} + \binom{a}{1}\binom{b}{n-1} + \dots + \binom{a}{n}\binom{b}{0} = \binom{a+b}{n}. \tag{12.9}$$

Prove this by induction. (*Hint*: Prove first that equation (12.9) holds for $a = 1$ and all b.)

9. *Continuation*. By a comparison of the coefficients of t^n on both sides of

$$(1 + t)^a(1 + t)^b = (1 + t)^{a+b} \tag{12.10}$$

prove more generally that (12.9) is true for arbitrary numbers a, b (and integral n).

10. Using equation (12.9), prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}. \tag{12.11}$$

11. Using equation (12.11), prove that

$$\sum_{\nu=0}^{n} \frac{(2n)!}{(\nu!)^2(n-\nu)!^2} = \binom{2n}{n}^2. \tag{12.12}$$

12. Prove that for integers $0 < a < b$

$$\sum_{k=1}^{a} (-1)^{a-k} \binom{a}{k}\binom{b+k}{b+1} = \binom{b}{a-1}. \tag{12.13}$$

Hint: Using (12.4) show that (12.11) is a special case of (12.9). Alternatively, compare coefficients of t^{a-1} in $(1-t)^a(1-t)^{-b-2} = (1-t)^{a-b-2}$.

13. By specialization derive from (12.9) the identities

$$\binom{a}{k} - \binom{a}{k-1} + - \dots \mp \binom{a}{1} \pm 1 = \binom{a-1}{k} \tag{12.14}$$

and

$$\sum_{\nu} (-1)^{\nu} \binom{a}{\nu}\binom{n-\nu}{r} = \binom{n-a}{n-r}, \tag{12.15}$$

valid if k, n, and r are positive integers. [*Hint:* Use (12.4).]

14. Using equation (12.9), prove that

$$\sum_{j=0}^{k} \binom{a+k-j-1}{k-j}\binom{b+j-1}{j} = \binom{a+b+k-1}{k}. \tag{12.16}$$

(*Hint:* Apply equation (12.4) back and forth.) Note the important special cases $b = 1, 2$.

15. Referring to the problems of section 11, notice that equations (11.12), (11.14), (11,15), and (11.16) define probabilities. In each the quantities should therefore add to unity. Show that this is implied, respectively, by (12.8), (12.9), (12.16), and the binomial theorem.

16. From the definition of $A(r, n)$ in problem 7 of section 11 it follows that $A(r, n) = 0$ if $r < n$ and $A(n, n) = n!$. In other words

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r = \begin{matrix} 0 & \text{if } r < n \\ n! & \text{if } r = n. \end{matrix} \tag{12.17}$$

(*a*) Prove (12.17) directly by reduction from n to $n - 1$. (*b*) Next prove (12.17) by considering the rth derivative of $(1 - e^t)^n$ at $t = 0$. (*c*) Generalize (12.17) by starting from (11.11) instead of (11.6).

17. If $0 \le N \le n$ prove by induction that for each integer $r \ge 0$

$$\sum_{\nu=0}^{N}(-1)^{\nu}\binom{N}{\nu}(n-\nu)_r = \binom{n-N}{r-N}r!. \tag{12.18}$$

(Note that the right-hand member vanishes when $r < N$ and when $r > n$.) Verify (12.18) by considering the rth derivative of $t^{n-N}(t-1)^N$ at $t = 1$.

18. Prove by induction (using the binomial theorem)

$$\binom{n}{1}\frac{1}{1} - \binom{n}{2}\frac{1}{2} + \ldots + (-1)^{n-1}\binom{n}{n}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}. \tag{12.19}$$

Verify (12.19) by integrating the identity $\sum_{0}^{n-1}(1-t)^{\nu} = \{1-(1-t)^n\}t^{-1}$.

19. Show that for any positive integer m

$$(x + y + z)^m = \sum \frac{m!}{a!b!c!}x^a y^b z^c \tag{12.20}$$

where the summation extends over all non-negative integers a, b, c, such that $a + b + c = m$.

20. Using Stirling's formula, prove that

$$\binom{2n}{n} \sim (\pi n)^{-\frac{1}{2}}2^{2n}. \tag{12.21}$$

21. Prove that for any positive integers a and b

$$\frac{(a+1)(a+2)\cdots(a+n)}{(b+1)(b+2)\cdots(b+n)} \sim \frac{b!}{a!}n^{a-b}. \tag{12.22}$$

22. The *gamma function* is defined by

$$\Gamma(x) = \int_0^{\infty} z^{x-1}e^{-z}\,dz \tag{12.23}$$

where $x > 0$. Show that $\Gamma(x) \sim (2\pi)^{\frac{1}{2}}e^{-x}x^{x-\frac{1}{2}}$. (Notice that if $x = n$ is an integer, $\Gamma(n) = (n-1)!$.)

23. Let a and r be arbitrary positive numbers and n a positive integer. Show that

$$a(a+r)(a+2r)\cdots(a+nr) \sim Cr^{n+1}n^{n+(a/r)+\frac{1}{2}}e^{-n}. \tag{12.24}$$

[The constant C is equal to $(2\pi)^{\frac{1}{2}}/\Gamma(a/r)$.]

24. Using the results of the preceding problem, show that

$$\frac{a(a+r)(a+2r)\cdots(a+nr)}{b(b+r)(b+2r)\cdots(b+nr)} \sim \frac{\Gamma(b/r)}{\Gamma(a/r)}\, n^{(a-b)/r}. \tag{12.25}$$

25. Prove the following *alternative form of Stirling's formula:*

$$n! \sim (2\pi)^{\frac{1}{2}}(n+\tfrac{1}{2})^{n+\frac{1}{2}}e^{-(n+\frac{1}{2})}. \tag{12.26}$$

26. *Continuation.* Using the method of the text, show that

$$(2\pi)^{\frac{1}{2}}(n+\tfrac{1}{2})^{n+\frac{1}{2}}e^{-(n+\frac{1}{2})-1/24(n+\frac{1}{2})} < n! < (2\pi)^{\frac{1}{2}}(n+\tfrac{1}{2})^{n+\frac{1}{2}}e^{-(n+\frac{1}{2})}. \tag{12.27}$$

27. Extending Stirling's formula, prove that

$$n! \sim (2\pi)^{\frac{1}{2}}n^{n+\frac{1}{2}}\exp\left\{-n + \frac{1}{12n} - \frac{1}{360n^3} + \cdots\right\}. \tag{12.28}$$