CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

In recent years, the growth of statistics has made itself felt in almost every phase of human activity. Statistics no longer consists merely of the collection of data and their presentation in charts and tables; it is now considered to encompass the science of basing inferences on observed data and the entire problem of making decisions in the face of uncertainty. This covers considerable ground since uncertainties are met when we flip a coin, when a dietician experiments with food additives, when an actuary determines life insurance premiums, when a quality control engineer accepts or rejects manufactured products, when a teacher compares the abilities of students, when an economist forecasts trends, when a newspaper predicts an election, and so forth.

It would be presumptuous to say that statistics, in its present state of development, can handle all situations involving uncertainties, but new techniques are constantly being developed and modern statistics can, at least, provide the framework for looking at these situations in a logical and systematic fashion. In other words, statistics provides the models that are needed to study situations involving uncertainties, in the same way as calculus provides the models that are needed to describe, say, the concepts of Newtonian physics.

The beginnings of the mathematics of statistics may be found in mid-eighteenth-century studies in probability motivated by interest in games of chance. The theory thus developed for "heads or tails" or "red or black" soon found applications in situations where the outcomes were "boy or girl," "life or death," or "pass or fail," and scholars began to apply probability theory to actuarial problems and some aspects of the social sciences. Later, probability and statistics were introduced into physics by L. Boltzmann, J. Gibbs, and J. Maxwell, and by this century they have found applications in all phases of human endeavor that in some way involve an element of uncertainty or risk. The names that are connected most prominently with the growth of mathematical statistics in the first half of the 20th century are those of

R. A. Fisher, J. Neyman, E. S. Pearson, and A. Wald. More recently, the work of R. Schlaifer, L. J. Savage, and others has given impetus to statistical theories based essentially on methods that date back to the eighteenth-century English clergyman Thomas Bayes.

The approach to statistical inference presented in this book is essentially the classical approach, with methods of inference based largely on the work of J. Neyman and E. S. Pearson. However, the more general decision-theory approach is introduced in Chapter 9 and some Bayesian methods are presented in Chapter 10. This material may be omitted without loss of continuity.

This book primarily is intended as a presentation of the *mathematical theory* underlying the modern practice of statistics. Mathematical statistics is a recognized branch of mathematics, and it can be studied for its own sake by students of mathematics. Today, the theory of statistics is applied to engineering, physics and astronomy, quality assurance and reliability, drug development, public health and medicine, the design of agricultural or industrial experiments, experimental psychology, and so forth. Those wishing to participate in such applications or to develop new applications will do well to understand the mathematical theory of statistics. For only through such an understanding can applications proceed without the serious mistakes that sometimes occur. The applications are illustrated by means of examples and a separate set of applied exercises, many of them involving the use of computers. To this end, we have added at the end of most chapters a discussion of how the theory of that chapter is applied in practice.

We begin with a brief review of combinatorial methods and binomial coefficients, giving material that we shall rely on in our forthcoming discussions of probability and probability distributions.

1.2 COMBINATORIAL METHODS

In many problems of statistics we must list all the alternatives that are possible in a given situation, or at least determine how many different possibilities there are. In connection with the latter, we often use the following theorem, sometimes called the **basic principle of counting,** the **counting rule for compound** events, or the **rule for the multiplication of choices.**

THEOREM **1.1.** If an operation consists of two steps, of which the first can be done in n_1 ways and for each of these the second can be done in n_2 ways, then the whole operation can be done in $n_1 \cdot n_2$ ways.

Here, "operation" stands for any kind of procedure, process, or method of selection.

To justify this theorem, let us define the ordered pair (x_i, y_j) to be the outcome that arises when the first step results in possibility x_i and the second step results in possibility y_j. Then, the set of all possible outcomes is composed of the following $n_1 \cdot n_2$ pairs:

$$
\begin{array}{c}
(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_{n_2}) \\
(x_2, y_1), (x_2, y_2), \ldots, (x_2, y_{n_2}) \\
\ldots \\
(x_{n_1}, y_1), (x_{n_1}, y_2), \ldots, (x_{n_1}, y_{n_2})
\end{array}
$$

EXAMPLE 1.1

Suppose that someone wants to go by bus, by train, or by plane on a week's vacation to one of the five East North Central States. Find the number of different ways in which this can be done.

Solution The particular state can be chosen in $n_1 = 5$ ways and the means of transportation can be chosen in $n_2 = 3$ ways. Therefore, the trip can be carried out in $5 \cdot 3 = 15$ possible ways. If an actual listing of all the possibilities is desirable, a **tree diagram** like that in Figure **1.1** provides a systematic approach. This diagram shows

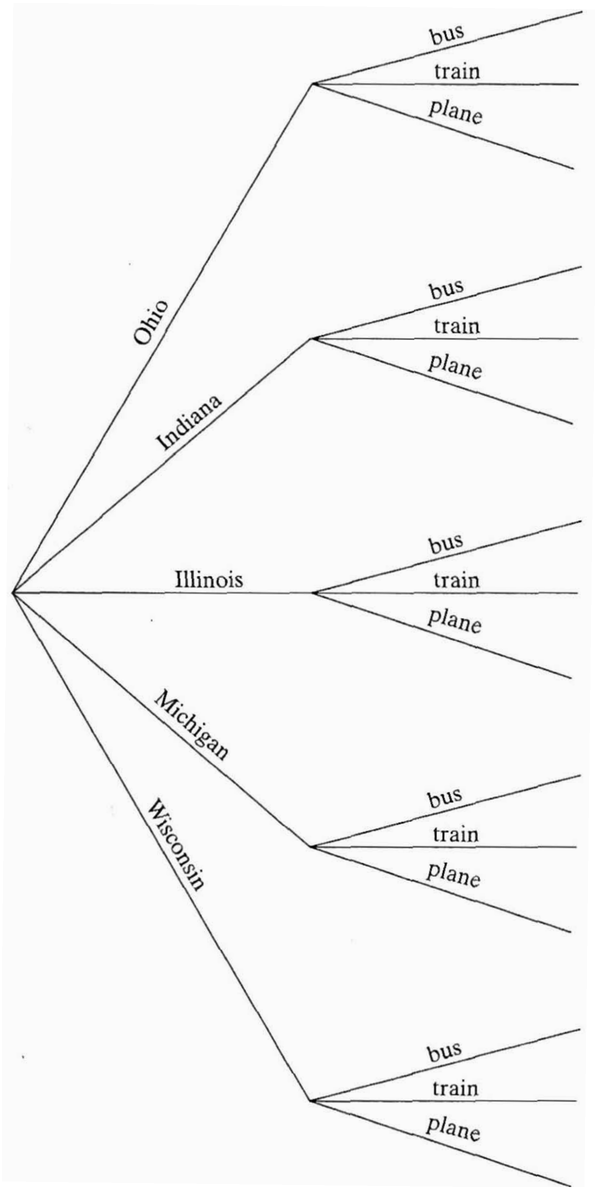

FIGURE 1.1: Tree diagram.

that there are $n_1 = 5$ branches (possibilities) for the number of states, and for each of these branches there are $n_2 = 3$ branches (possibilities) for the different means of transportation. It is apparent that the 15 possible ways of taking the vacation are represented by the 15 distinct paths along the branches of the tree. ■

EXAMPLE 1.2

How many possible outcomes are there when we roll a pair of dice, one red and one green?

Solution The red die can land in any one of six ways, and for each of these six ways the green die can also land in six ways. Therefore, the pair of dice can land in $6 \cdot 6 = 36$ ways. ■

Theorem 1.1 may be extended to cover situations where an operation consists of two or more steps. In this case,

THEOREM 1.2. If an operation consists of k steps, of which the first can be done in n_1 ways, for each of these the second step can be done in n_2 ways, for each of the first two the third step can be done in n_3 ways, and so forth, then the whole operation can be done in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways.

EXAMPLE 1.3

A quality control inspector wishes to select a part for inspection from each of four different bins containing 4, 3, 5, and 4 parts, respectively. In how many different ways can she choose the four parts?

Solution The total number of ways is $4 \cdot 3 \cdot 5 \cdot 4 = 240$. □

EXAMPLE 1.4

In how many different ways can one answer all the questions of a true-false test consisting of 20 questions?

Solution Altogether there are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2 = 2^{20} = 1{,}048{,}576$$

different ways in which one can answer all the questions; only one of these corresponds to the case where all the questions are correct and only one corresponds to the case where all the answers are wrong. □

Frequently, we are interested in situations where the outcomes are the different ways in which a group of objects can be ordered or arranged. For instance, we might want to know in how many different ways the 24 members of a club can elect a president, a vice president, a treasurer, and a secretary, or we might want to know in how many different ways six persons can be seated around a table. Different arrangements like these are called **permutations.**

EXAMPLE 1.5

How many permutations are there of the letters a, b, and c?

Solution The possible arrangements are abc, acb, *bac*, bca, cab, and cba, so the number of distinct permutations is six. Using Theorem 1.2, we could have arrived at this answer without actually listing the different permutations. Since there are three choices to select a letter for the first position, then two for the second position, leaving only one letter for the third position, the total number of permutations is $3 \cdot 2 \cdot 1 = 6$. ■

Generalizing the argument used in the preceding example, we find that n distinct objects can be arranged in $n(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$ different ways. To simplify our notation, we represent this product by the symbol n!, which is read "n factorial." Thus, $1! = 1$, $2! = 2.1 = 2$, $3! = 3\cdot 2\cdot 1 = 6$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$, $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$, and so on. Also, by definition we let $0! = 1$.

THEOREM 1.3. The number of permutations of n distinct objects is n!.

EXAMPLE 1.6

In how many different ways can the five starting players of a basketball team be introduced to the public?

Solution There are $5! = 5 \cdot 4 . 3 \cdot 2 \cdot 1 = 120$ ways in which they can be introduced. ■

EXAMPLE 1.7

The number of permutations of the four letters a, b, c, and d is 24, but what is the number of permutations if we take only two of the four letters or, as it is usually put, if we take the four letters two at a time?

Solution We have two positions to fill, with four choices for the first and then three choices for the second. Therefore, by Theorem 1.1, the number of permutations is $4 \cdot 3 = 12$. ■

Generalizing the argument that we used in the preceding example, we find that n distinct objects taken r at a time, for $r > 0$, can be arranged in $n(n-1)\cdot\ldots\cdot(n-r+1)$ ways. We denote this product by ${}_nP_r$, and we let ${}_nP_0 = 1$ by definition. Therefore, we can write

THEOREM 1.4. The number of permutations of n distinct objects taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

for $r = 0, 1, 2, \ldots, n$.

Proof. The formula ${}_nP_r = n(n-1)\cdot \ldots \cdot (n-r+1)$ cannot be used for $r = 0$, but we do have

$$ {}_nP_0 = \frac{n!}{(n-0)!} = 1 $$

For $r = 1, 2, \ldots, n$, we have

$$ \begin{aligned} {}_nP_r &= n(n-1)(n-2)\cdot \ldots \cdot (n-r-1) \\ &= \frac{n(n-1)(n-2)\cdot \ldots \cdot (n-r-1)(n-r)!}{(n-r)!} \\ &= \frac{a!}{(n-r)!} \end{aligned} $$

In problems concerning permutations, it is usually easier to proceed by using Theorem 1.2 as in Example 1.7, but the factorial formula of Theorem 1.4 is somewhat easier to remember. Many statistical software packages provide values of ${}_nP_r$ and other combinatorial quantities upon simple commands. Indeed, these quantities are also preprogrammed in many hand-held statistical (or scientific) calculators.

EXAMPLE 1.8

Four names are drawn from among the 24 members of a club for the offices of president, vice president, treasurer, and secretary. In how many different ways can this be done?

Solution The number of permutations of 24 distinct objects taken four at a time is

$$ {}_{24}P_4 = \frac{24!}{20!} = 24\cdot 23\cdot 22\cdot 21 = 255{,}024 $$

EXAMPLE 1.9

In how many ways can a local chapter of the American Chemical Society schedule three speakers for three different meetings if they are all available on any of five possible dates?

Solution Since we must choose three of the five dates and the order in which they are chosen (assigned to the three speakers) matters, we get

$$ {}_5P_3 = \frac{5!}{2!} = \frac{120}{2} = 60 $$

We might also argue that the first speaker can be scheduled in five ways, the second speaker in four ways, and the third speaker in three ways, so that the answer is $5\cdot 4\cdot 3 = 60$. ■

Permutations that occur when objects are arranged in a circle are called **circular permutations.** Two circular permutations are not considered different (and are counted only once) if corresponding objects in the two arrangements have the same objects to their left and to their right. For example, if four persons are playing bridge, we do not get a different permutation if everyone moves to the chair at his or her right.

EXAMPLE 1.10

How many circular permutations are there of four persons playing bridge?

Solution If we arbitrarily consider the position of one of the four players as fixed, we can seat (arrange) the other three players in $3! = 6$ different ways. In other words, there are six different circular permutations. ■

Generalizing the argument used in the preceding example, we get the following theorem.

THEOREM 1.5. The number of permutations of n distinct objects arranged in a circle is $(n-1)!$.

We have been assuming until now that the n objects from which we select r objects and form permutations are all distinct. Thus, the various formulas cannot be used, for example, to determine the number of ways in which we can arrange the letters in the word "book," or the number of ways in which three copies of one novel and one copy each of four other novels can be arranged on a shelf.

EXAMPLE 1.11

How many different permutations are there of the letters in the word "book"?

Solution If we distinguish for the moment between the two o's by labeling them o_1 and o_2, there are $4! = 24$ different permutations of the symbols b, o_1, o_2, and k. However, if we drop the subscripts, then bo_1ko_2 and bo_2ko_1, for instance, both yield boko, and since each pair of permutations with subscripts yields but one arrangement without subscripts, the total number of arrangements of the letters in the word "book" is $\frac{24}{2} = 12$. ■

EXAMPLE 1.12

In how many different ways can three copies of one novel and one copy each of four other novels be arranged on a shelf?

Solution If we denote the three copies of the first novel by a_1, a_2, and a_3 and the other four novels by b, c, d, and e, we find that with subscripts there are 7! different permutations of a_1, a_2, a_3, b, c, d, and e. However, since there are 3! permutations of a_1, a_2, and a_3 that lead to the same permutation of a, a, a, b, c, d, and e, we find that there are only $\frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ ways in which the seven books can be arranged on a shelf. ■

Generalizing the argument that we used in the two preceding examples, we get the following theorem.

THEOREM 1.6. The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, ..., n_k are of a kth kind, and $n_1 + n_2 + \cdots + n_k = n$ is

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

EXAMPLE 1.13

In how many ways can two paintings by Monet, three paintings by Renoir, and two paintings by Degas be hung side by side on a museum wall if we do not distinguish between the paintings by the same artists?

Solution Substituting $n = 7, n_1 = 2, n_2 = 3$, and $n_3 = 2$ into the formula of Theorem 1.6, we get

$$\frac{7!}{2! \cdot 3! \cdot 2!} = 210$$

■

There are many problems in which we are interested in determining the number of ways in which r objects can be selected from among n distinct objects *without regard to the order in which they are selected.* Such selections (arrangements) are called **combinations.**

EXAMPLE 1.14

In how many different ways can a person gathering data for a market research organization select three of the 20 households living in a certain apartment complex?

Solution If we care about the order in which the households are selected, the answer is

$$_{20}P_3 = 20 \cdot 19 \cdot 18 = 6{,}840$$

but each set of three households would then be counted $3! = 6$ times. If we do not care about the order in which the households are selected, there are only $\frac{6{,}840}{6} = 1{,}140$ ways in which the person gathering the data can do his or her job.

■

Actually, "combination" means the same as "subset," and when we ask for the number of combinations of r objects selected from a set of n distinct objects, we are simply asking for the total number of subsets of r objects that can be selected from a set of n distinct objects. In general, there are $r!$ permutations of the objects in a subset of r objects, so that the. P_r permutations of r objects selected from a set of n distinct objects contain each subset $r!$ times. Dividing $_nP_r$ by $r!$ and denoting the result by the symbol $\binom{n}{r}$, we thus have

THEOREM 1.7. The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

for $r = 0, 1, 2, \ldots, n$.

EXAMPLE 1.15

In how many different ways can six tosses of a coin yield two heads and four tails?

Solution This question is the same as asking for the number of ways in which we can select the two tosses on which heads is to occur. Therefore, applying Theorem 1.7, we find that the answer is

$$\binom{6}{2} = \frac{6!}{2! \cdot 4!} = 15$$

This result could also have been obtained by the rather tedious process of enumerating the various possibilities, HHTTTT, TTHTHT, HTHTTT, . . . , where H stands for head and T for tail. ■

EXAMPLE 1.16

How many different committees of two chemists and one physicist can be formed from the four chemists and three physicists on the faculty of a small college?

Solution Since two of four chemists can be selected in $\binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6$ ways and one of three physicists can be selected in $\binom{3}{1} = \frac{3!}{1! \cdot 2!} = 3$ ways, Theorem 1.1 shows that the number of committees is $6 \cdot 3 = 18$. ■

A combination of r objects selected from a set of n distinct objects may be considered a **partition** of the n objects into two subsets containing, respectively, the r objects that are selected and the $n - r$ objects that are left. Often, we are concerned with the more general problem of partitioning a set of n distinct objects into k subsets, which requires that each of the n objects must belong to one and only one of the subsets.† The order of the objects within a subset is of no importance.

EXAMPLE 1.17

In how many ways can a set of four objects be partitioned into three subsets containing, respectively, two, one, and one of the objects?

Solution Denoting the four objects by a, b, c, and d, we find by enumeration that there are the following 12 possibilities:

$$\begin{array}{llll} ab|c|d & ab|d|c & ac|b|d & ac|d|b \\ ad|b|c & ad|c|b & bc|a|d & bc|d|a \\ bd|a|c & bd|c|a & cd|a|b & cd|b|a \end{array}$$

The number of partitions for this example is denoted by the symbol

$$\binom{4}{2, 1, 1} = 12$$

---

†Symbolically, the subsets $A_1, A_2, \ldots, A_k$ constitute a partition of set A if $A_1 \cup A_2 \cup \cdots \cup A_k = A$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

where the number at the top represents the total number of objects and the numbers at the bottom represent the number of objects going into each subset. ■

Had we not wanted to enumerate all the possibilities in the preceding example, we could have argued that the two objects going into the first subset can be chosen in $\binom{4}{2} = 6$ ways, the object going into the second subset can then be chosen in $\binom{2}{1} = 2$ ways, and the object going into the third subset can then be chosen in $\binom{1}{1} = 1$ way. Thus, by Theorem 1.2 there are $6 \cdot 2 \cdot 1 = 12$ partitions. Generalizing this argument, we have the following theorem.

THEOREM **1.8.** The number of ways in which a set of n distinct objects can be partitioned into k subsets with n_1 objects in the first subset, n_2 objects in the second subset, . . . , and n_k objects in the kth subset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

Proof. Since the n_1 objects going into the first subset can be chosen in $\binom{n}{n_1}$ ways, the n_2 objects going into the second subset can then be chosen in $\binom{n-n_1}{n_2}$ ways, the n_3 objects going into the third subset can then be chosen in $\binom{n-n_1-n_2}{n_3}$ ways, and so forth, it follows by Theorem 1.2 that the total number of partitions is

$$\begin{aligned}
\binom{n}{n_1, n_2, \ldots, n_k} &= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \ldots \cdot \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\
&= \frac{n!}{n\,!\,(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \\
&\quad \cdot \ldots \cdot \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k! \cdot 0!} \\
&= \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\end{aligned}$$

EXAMPLE 1.18

In how many ways can seven businessmen attending a convention be assigned to one triple and two double hotel rooms?

Solution Substituting $n = 7, n_1 = 3, n_2 = 2$, and $n_3 = 2$ into the formula of Theorem 1.8, we get

$$\binom{7}{3,\ 2,\ 2} = \frac{7!}{3! \cdot 2! \cdot 2!} = 210$$

1.3 BINOMIAL COEFFICIENTS

If n is a positive integer and we multiply out $(x + y)^n$ term by term, each term will be the product of x's and y's, with an x or a y coming from each of the n factors x + y. For instance, the expansion

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= x\cdot x\cdot x + x\cdot x\cdot y + x\cdot y\cdot x + x\cdot y\cdot y\\ &\quad + y\cdot x\cdot x + y\cdot x\cdot y + y\cdot y\cdot x + y\cdot y\cdot y\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

yields terms of the form x^3, x^2y, xy^2, and y^3. Their coefficients are 1, 3, 3, and 1, and the coefficient of xy^2, for example, is $\binom{3}{2} = 3$, the number of ways in which we can choose the two factors providing the y's. Similarly, the coefficient of x^2y is $\binom{3}{1} = 3$, the number of ways in which we can choose the one factor providing the y, and the coefficients of x^3 and y^3 are $\binom{3}{0} = 1$ and $\binom{3}{3} = 1$.

More generally, if n is a positive integer and we multiply out $(x + y)^n$ term by term, the coefficient of $x^{n-r}y^r$ is $\binom{n}{r}$, the number of ways in which we can choose the r factors providing the y's. Accordingly, we refer to $\binom{n}{r}$ as a **binomial coefficient.** We can now state the following theorem.

THEOREM 1.9.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \quad \text{for any positive integer } n$$

(For readers who are not familiar with the $\sum$ notation, a brief explanation is given in Appendix A.)

The calculation of binomial coefficients can often be simplified by making use of the three theorems that follow.

THEOREM 1.10. For any positive integers n and $r = 0, 1, 2, \ldots, n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof. We might argue that when we select a subset of r objects from a set of n distinct objects, we leave a subset of $n - r$ objects; hence, there are as many ways of selecting r objects as there are ways of leaving (or selecting) $n - r$ objects. To prove the theorem algebraically, we write

$$\begin{aligned}\binom{n}{n-r} &= \frac{n!}{\text{n} \quad - \quad \text{r} \quad -} = \frac{n!}{\text{n}\ (n-r)!r!}\\ &= \frac{n!}{r!(n-r)!} = \binom{n}{r}\end{aligned}$$

□

Theorem 1.10 implies that if we calculate the binomial coefficients for $r = 0, 1, \ldots, \frac{n}{2}$ when n is even and for $r = 0, 1, \ldots, \frac{n-1}{2}$, when a is odd, the remaining binomial coefficients can be obtained by making use of the theorem.

EXAMPLE 1.19

Given $\binom{4}{0} = 1, \binom{4}{1} = 4$, and $\binom{4}{2} = 6$, find $\binom{4}{3}$ and $\binom{4}{4}$.

Solution

$$\binom{4}{3} = \binom{4}{4-3} = \binom{4}{1} = 4 \text{ and } \binom{4}{4} = \binom{4}{4-4} = \binom{4}{0} = 1$$

EXAMPLE 1.20

Given $\binom{5}{0} = 1, \binom{5}{1} = 5$, and $\binom{5}{2} = 10$, find $\binom{5}{3}, \binom{5}{4}$, and $\binom{5}{5}$

Solution

$$\binom{5}{3} = \binom{5}{5-3} = \binom{5}{2} = 10, \binom{5}{4} = \binom{5}{5-4} = \binom{5}{1} = 5, \text{ and}$$

$$\binom{5}{5} = \binom{5}{5-5} = \binom{5}{0} = 1$$

It is precisely in this fashion that Theorem 1.10 may have to be used in connection with Table VII.[†]

EXAMPLE 1.21

Find $\binom{20}{12}$ and $\binom{17}{10}$

Solution Since $\binom{20}{12}$ is not given in Table VII, we make use of the fact that $\binom{20}{12} = \binom{20}{8}$, look up $\binom{20}{8}$, and get $\binom{20}{12} = 125.970$. Similarly, to find $\binom{17}{10}$, we make use of the fact that $\binom{17}{10} = \binom{17}{7}$, look up $\binom{17}{7}$, and get $\binom{17}{10} = 19{,}448$. ■

THEOREM **1.11.** For any positive integer n and $r = 1, 2, \ldots, n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

[†]Roman numerals refer to the statistical tables at the end of the book.

Proof. Substituting $x = 1$ into $(x+y)^n$, let us write $(1+y)^n = (1+y)(1+y)^{n-1} = (1+y)^{n-1} + y(1+y)^{n-1}$ and equate the coefficient of y^r in $(1+y)^n$ with that in $(1+y)^{n-1} + y(1+y)^{n-1}$. Since the coefficient of y^r in $(1+y)^n$ is $\binom{n}{r}$ and the coefficient of y^r in $(1+y)^{n-1} + y(1+y)^{n-1}$ is the sum of the coefficient of y^r in $(1+y)^{n-1}$, that is, $\binom{n-1}{r}$, and the coefficient of y^{r-1} in $(1+y)^{n-1}$, that is, $\binom{n-1}{r-1}$, we obtain

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

which completes the proof. □

Alternatively, take any one of the n objects. If it is not to be included among the r objects, there are $\binom{n-1}{r}$ ways of selecting the r objects; if it is to be included, there are $\binom{n-1}{r-1}$ ways of selecting the other $r-1$ objects. Therefore, there are $\binom{n-1}{r} + \binom{n-1}{r-1}$ ways of selecting the r objects, that is,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Theorem 1.11 can also be proved by expressing the binomial coefficients on both sides of the equation in terms of factorials and then proceeding algebraically, but we shall leave this to the reader in Exercise 1.12. An important application of Theorem 1.11 is a construct known as **Pascal's triangle.** When no table is available, it is sometimes convenient to determine binomial coefficients by means of a simple construction. Applying Theorem 1.11, we can generate Pascal's triangle as follows:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\multicolumn{11}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

In this triangle, the first and last entries of each row are the numeral "1" each other entry in any given row is obtained by adding the two entries in the preceding row immediately to its left and to its right.

To state the third theorem about binomial coefficients, let us make the following definition: $\binom{n}{r} = 0$ whenever n is a positive integer and r is a positive integer greater than n. (Clearly, there is no way in which we can select a subset that contains more elements than the whole set itself.)

THEOREM 1.12.

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$$

Proof. Using the same technique as in the proof of Theorem 1.11, let us prove this theorem by equating the coefficients of y^k in the expressions on both sides of the equation

$$(1+y)^{m+n} = (1+y)^m(1+y)^n$$

The coefficient of y^k in $(1+y)^{m+n}$ is $\binom{m+n}{k}$, and the coefficient of y^k in

$$\begin{aligned}(1+y)^m(1+y)^n &= \left[\binom{m}{0}+\binom{m}{1}y+\cdots+\binom{m}{m}y^m\right]\\ &\quad\times\left[\binom{n}{0}+\binom{n}{1}y+\cdots+\binom{n}{n}y^n\right]\end{aligned}$$

is the sum of the products that we obtain by multiplying the constant term of the first factor by the coefficient of y^k in the second factor, the coefficient of y in the first factor by the coefficient of y^{k-1} in the second factor, ..., and the coefficient of y^k in the first factor by the constant term of the second factor. Thus, the coefficient of y^k in $(1+y)^m(1+y)^n$ is

$$\begin{aligned}&\binom{m}{0}\binom{n}{k}+\binom{m}{1}\binom{n}{k-1}+\binom{m}{2}\binom{n}{k-2}+\cdots+\binom{m}{k}\binom{n}{0}\\ &=\sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r}\end{aligned}$$

and this completes the proof.

EXAMPLE 1.22

Verify Theorem 1.12 numerically for $m=2$, $n=3$, and $k=4$.

Solution Substituting these values, we get

$$\binom{2}{0}\binom{3}{4}+\binom{2}{1}\binom{3}{3}+\binom{2}{2}\binom{3}{2}+\binom{2}{3}\binom{3}{1}+\binom{2}{4}\binom{3}{0}=\binom{5}{4}$$

and since $\binom{3}{4}$, $\binom{2}{3}$, and $\binom{2}{4}$ equal 0 according to the definition on page 13, the equation reduces to

$$\binom{2}{1}\binom{3}{3}+\binom{2}{2}\binom{3}{2}=\binom{5}{4}$$

which checks, since $2\cdot 1+1\cdot 3=5$. ■

Using Theorem 1.8, we can extend our discussion to **multinomial coefficients,** that is, to the coefficients that arise in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$. The multinomial coefficient of the term $x_1^{r_1} \cdot x_2^{r_2} \cdot \ldots \cdot x_k^{r_k}$ in the expansion of $(x_1 + x_2 + \ldots + x_k)^n$ is

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_k!}$$

EXAMPLE 1.23

What is the coefficient of $x_1^3 x_2 x_3^2$ in the expansion of $(x_1 + x_2 + x_3)^6$?

Solution Substituting $n = 6, r_1 = 3, r_2 = 1$, and $r_3 = 2$ into the preceding formula, we get

$$\frac{6!}{3! \cdot 1! \cdot 2!} = 60$$

EXERCISES

1.1. An operation consists of two steps, of which the first can be made in n_1 ways. If the first step is made in the ith way, the second step can be made in n_{2i} ways.†

(a) Use a tree diagram to find a formula for the total number of ways in which the total operation can be made.

(b) **A** student can study 0, 1, 2, or 3 hours for a history test on any given day. Use the formula obtained in part (a) to verify that there are 13 ways in which the student can study at most 4 hours for the test on two consecutive days.

1.2. With reference to Exercise 1.1, verify that if n_{2i} equals the constant n_2, the formula obtained in part (a) reduces to that of Theorem 1.1.

1.3. With reference to Exercise 1.1, suppose that there is a third step, and if the first step is made in the ith way and the second step in the jth way, the third step can be made in n_{3ij} ways.

(a) Use a tree diagram to verify that the whole operation can be made in

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_{2i}} n_{3ij}$$

different ways.

(b) With reference to part (b) of Exercise 1.1, use the formula of part (a) to verify that there are 32 ways in which the student can study at most 4 hours for the test on three consecutive days.

1.4. Show that if n_{2i} equals the constant n_2 and n_{3ij} equals the constant n_3, the formula of part (a) of Exercise 1.3 reduces to that of Theorem 1.2.

1.5. In a two-team basketball play-off, the winner is the first team to win m games.

(a) Counting separately the number of play-offs requiring $m, m + 1, \ldots$, and $2m - 1$ games, show that the total number of different outcomes (sequences of wins and losses by one of the teams) is

$$2\left[\binom{m-1}{m-1} + \binom{m}{m-1} + \cdots + \binom{2m-2}{m-1}\right]$$

†The use of double subscripts is explained in Appendix **A.**