Course: CS5961/6951 *Computational Statistics* Sp 2011
Instructor: R. F. Riesenfeld
Date: 15 Mar 2011
Due: Fri, 18 Jan 2010

Notes: *Poisson Distribution as the Limit of Binomial Distribution*

Consider a small (Δt) interval in a larger interval $I = [I_a, I_b]$ of the real line, so that $I = N(\Delta t)$, for some large N>0 . In other words, $(\Delta t) = \frac{I}{N}$, so there are N intervals in I .

Now, suppose that $P\{x \in (\Delta t)_i\} = \lambda(\Delta t)_i$, where λ is a normalizing constant $\left(= \frac{1}{I}\right)$. We can think of λ as linear probability density function, analogous to computing the total resistance of a wire as ρL, where ρ is a constant giving unit length resistance and L is the overall length of the wire under consideration.

From the Binomial Probability Distribution, we know how to compute the likelihood of $k\ successes\ in\ N\ Bernoulli\ trials$. Specifically,

$P\{k\ successes\ in\ N\ trials\} = \binom{N}{k} q^{N-k} p^k$, where p is the probability of success in a Bernoulli trial and q is the probability of failure. However, here we know that $p = \frac{\lambda I}{N}$ and $q = 1 - p$. Hence,

$$P\{k\ successes\ in\ N\ trials\} = \frac{N!}{(N-k)k!}\left(1 - \frac{\lambda I}{N}\right)^{N-k} \frac{\lambda I}{N}^{k}$$

$$= \frac{(\lambda I)^k}{k!}\ N(N-1) \cdot \ldots \cdot (N-k-1)\left(1 - \frac{\lambda I}{N}\right)^{N-k} \left(\frac{1}{N}\right)^k$$

At this point, noting that $\lambda, I\ and\ k$ are fixed, we take the limit of the above equation as $N \rightarrow \infty$. Thus,

$$\lim_{N\rightarrow\infty} P\{k\ successes\ in\ N\ trials\}$$

$$= \lim_{N\rightarrow\infty}\left(\frac{(\lambda I)^k}{k!}\ N(N-1)\cdot\ldots\cdot(N-k-1)\left(1-\frac{\lambda I}{N}\right)^{N-k}\left(\frac{1}{N}\right)^k\right)$$

$$= \frac{(\lambda I)^k}{k!}\lim_{N\rightarrow\infty}\left(\ N(N-1)\cdot\ldots\cdot(N-k-1)\left(1-\frac{\lambda I}{N}\right)^{N-k}\left(\frac{1}{N}\right)^k\right)\ .$$

And, with some slight rearranging, the following is arrived at,

$$= \frac{(\lambda I)^k}{k!}\lim_{N\rightarrow\infty}\left(\frac{N}{N}\frac{(N-1)}{N}\cdot\ldots\cdot\frac{(N-k-1)}{N}\left(1-\frac{\lambda I}{N}\right)^{N-k}\right)$$

$$= \frac{(\lambda I)^k}{k!}\lim_{N\rightarrow\infty}\left(\frac{N}{N}\frac{(N-1)}{N}\cdot\ldots\cdot\frac{(N-k-1)}{N}\left(1-\frac{\lambda I}{N}\right)^{N}\left(1-\frac{\lambda I}{N}\right)^{-k}\right)\ .$$

First, we make the following trivial observations about the two limits,

$$\lim_{N\rightarrow\infty}\left(\frac{N}{N}\frac{(N-1)}{N}\cdot\ldots\cdot\frac{(N-k-1)}{N} = 1\right)$$

and,

$$\lim_{N\rightarrow\infty}\left(1-\frac{\lambda I}{N}\right)^{N} = 1\ \ .$$

Therefore, the subject limit reduces to the simpler expression,

$$= \frac{(\lambda I)^k}{k!} \lim_{N\to\infty} \left(1 - \frac{\lambda I}{N}\right)^N .$$

But, by the definition e given in Calculus, namely, $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{N}\right)^N$, we have only to observe that the desired limit immediately becomes,

$$= \frac{(\lambda I)^k}{k!} e^{-\lambda I} \quad .$$

Substituting the variable t for the arbitrary interval t leads use to the more common equivalent expression,

$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad .$$

The derived expression is known as the *Poisson Probability Distribution Function for k events, given λ as the mean number successes in unit measure (i.e., time, area, or the like).*

□