

**Lecture Notes:** Binomial distribution: expectation and variance

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**Definitions:**

$$b(k; n, p) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = E(X) = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k}$$

$$\sigma^2 = \text{Var}(X) = \sum_{k=0}^n k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

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1. As a preliminary step, we observe that the first term in the summation is annihilated by the factor  $k$ , because  $k=0$ . Therefore, we can drop the first term, and simply begin the summation at  $k=1$ ,

$$\begin{aligned} E(X) &= \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

Recalling the recursive relationship for binomial coefficients  $k \binom{n}{k} = n \binom{n-1}{k-1}$ , we use it to make a substitution,

$$= \sum_{k=1}^n n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

Now, since  $n$  and  $p$  are constants, we can pull them out of the summation operator, which yields,

$$= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

Note that the exponent of  $p^{k-1}$  above is reduced by 1 to reflect a factor of  $p$  has been factored outside the summation.

Changing the summation index to  $j=k-1$ , we can sum from 0 to  $n-1$  instead of 1 to  $n$ ,

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-j-1}$$

If we observe that the exponent  $n-j-1 = ((n-1)-j)$ , it is clear that the summation expression is now in canonical form for  $\sum b(j; n-1, p)$ . That is, the summation takes place over all possible outcomes  $j$  for the binomial distribution governing  $(n-1)$  Bernoulli trials. Therefore, the summation must simply equal 1, and we now can conclude that,

$$\boxed{\mu = E(X) = np}$$

This coincides with our intuition on how many heads we ought to expect, on average, in  $n$  tosses of a biased coin with probability of success  $p$ .

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2. To calculate the variance of the binomial distribution  $b(k; n, p)$ , we use the convenient formula,

$$\sigma^2 = Var(X) = E(X^2) - \mu^2$$

We have already derived the value of  $\mu$  above, so we now only need to consider  $E(X^2)$ ,

$$\begin{aligned} E(X^2) &= \sum_{k=0}^n k^2 \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

Again, the recursive formula for binomial coefficients  $k \binom{n}{k} = n \binom{n-1}{k-1}$  is used to make a substitution, and we observe, again, that the first term of the summation is 0 because  $k=0$  annihilates it. This leads to the following,

$$= \sum_{k=1}^n k \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

In a manner analogous to part 1 where we derived  $E(X)$ , we move  $n$  outside the summation because it is a constant, and we factor out a  $p$  from  $p^k$ , thus,

$$= np \sum_{k=1}^n k \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

Also we perform a similar change of variable  $j=k-1$  on the summation index, in a manner analogous to what we did in part 1, so we now have,

$$= np \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^j (1-p)^{(n-1)-j}$$

Distributing the summation over  $(j+1)$ , we generate two summations,

$$= np \left( \sum_{j=0}^{n-1} j \binom{n-1}{j} p^j (1-p)^{(n-1)-j} + \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j} \right)$$

This quickly simplifies if we observe that the second summation is 1 because it is a summation over all possibilities, i.e.,  $\sum b(j; n-1, p)$ , and the first summation is exactly the expectation of the binomial distribution for  $(n-1)$  Bernoulli trials. From part 1, we already have established that the first summation is  $p(n-1)$ . Thus, we now have,

$$E(X^2) = np((n-1)p + 1)$$

Returning to the earlier formula,

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu^2 \\ &= np((n-1)p + 1) - \mu^2 \end{aligned}$$

Recall that  $\mu = np$ ,

$$\begin{aligned} &= np((n-1)p + 1) - n^2 p^2 \\ &= n^2 p^2 - np^2 + np - n^2 p^2 \\ &= np - np^2 \end{aligned}$$

$$= np(1 - p)$$

$$\boxed{\text{Var}(X) = npq}$$

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