

Course: CS 5961/6951
Instructor: R. F. Riesenfeld
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Computational Statistics Spring 2009

Homework: #1 Combinatorics

Due: 19 Feb 2009

Problem 1: Assume knowledge that the formula $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$ is the number of

distinct subsets of k elements that can be formed from a pool of n elements, where $k \leq n$. Intuitively and convincingly illustrate in a diagrammatical manner, i.e., illustrate with objects, sets and subsets, a *counting argument* to persuade someone that the following recursive binomial coefficient relationship is correct,

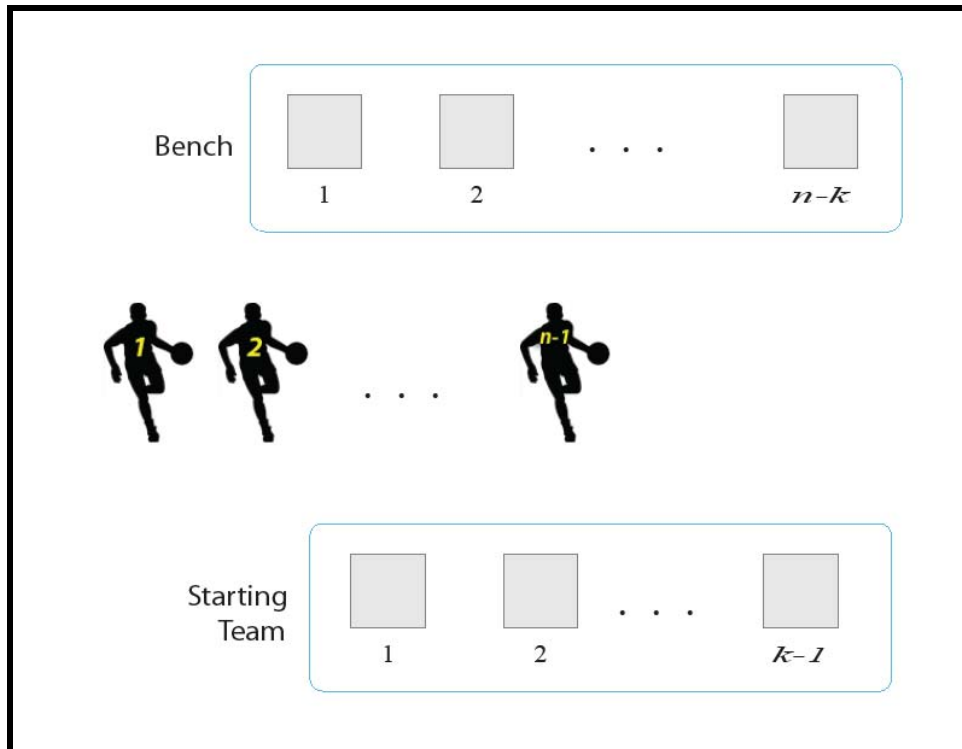
$$k \binom{n}{k} = n \binom{n-1}{k-1} \quad (1.1)$$

This was derived algebraically in class, but that does not provide concrete insight for why this must be true. We want to gain greater understanding for why the relationship obtains.

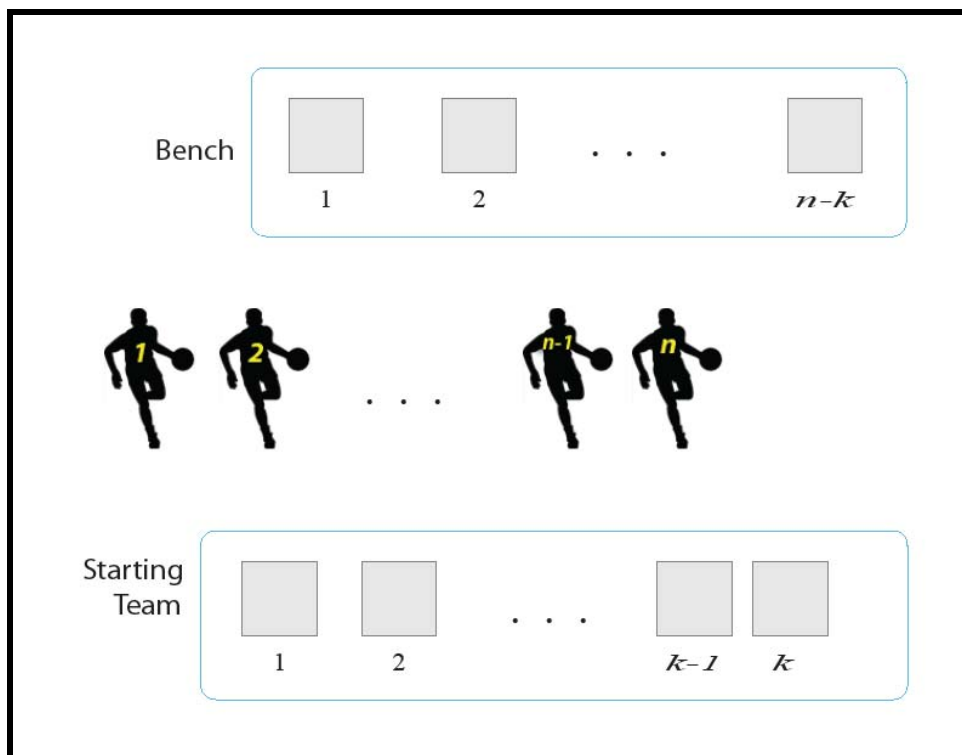
In class it was suggested to reason about a scenario in which a coach has a roster of $(n-1)$ players for a sport requiring $(k-1)$ players on a (starting) team. The coach's assistant has prepared an exhaustive list \mathcal{L} enumerating all such possible teams. The right hand side of the relationship pertains to this setting.

Further suppose that a new rule increases the size of teams to k from $(k-1)$, therefore the roster is also allowed to increase by 1 player. We wish to extend the analysis of the existing list \mathcal{L}_{n-1} of all teams of $(k-1)$ players and a roster of $(n-1)$ players to generate a list \mathcal{L}_n enumerating all possible teams of k players taken from a roster of n players.

- i) Use the formula for combinations to determine how many teams of $(k-1)$ players can be created from a roster of $(n-1)$ players? The below sketch illustrates the situation,



ii) Now, due to a revision in rules, add 1 more player to the roster because the team is now allowed k players, instead of $(k - 1)$. This is illustrated below.



- a. Recall, we have the list \mathcal{L}_{n-1} with teams of $(k-1)$ players. It is trivial to create starting teams of k players simply by adding the *player n* to make all the teams on list \mathcal{L}_{n-1} now have k players. Call the new list thusly created \mathcal{L}_n . How many teams are made by augmenting each team on list \mathcal{L}_{n-1} by adding *player n* as a starting player, as it were? That is, at this point, how many teams are there on \mathcal{L}_n ?
- b. But there are more teams of k players; teams based on \mathcal{L}_{n-1} are just a part of the full list. Now consider how many other distinct teams of k players can be created when the 1 extra *player n*, does not start. That is, *player n* sits on the bench. How many teams of k players can there be with a roster of n players, given that new *player n* must sit on the bench? [See figure in ii)]
- c. Having counted all teams of k players that include *player n* (iia, above), and all teams of k players with *player n* on the bench (iib, above), we can simply add the two expressions to get the total number of teams of k players. Carry out this calculation to get equation (1.1).

Problem 2: With reference to notes on mean and variance of the binomial distribution, derive the *Third Moment* of the binomial distribution:

$$\sum_{k=0}^n k^3 \binom{n}{k} p^k (1-p)^{n-k}$$