

Transformations II

CS5600 *Computer Graphics*
 From Rich Riesenfeld
 Spring 2013

Spring 2013 Utah School of Computing

Arbitrary 3D Rotation

- What is its inverse?
- What is its transpose?
- Can we constructively elucidate this relationship?

Spring 2013 Utah School of Computing 2

Rotate $+\theta$ about axis a : $R_a(+\theta)$

Spring 2013 Utah School of Computing 3

First, Rotate about z by $+\alpha$: $R_z(+\alpha)$

Spring 2013 Utah School of Computing 4

Then Rotate about x by $+\beta$: $R_x(+\beta)$

Spring 2013 Utah School of Computing 5

Now, $+\theta$ Rotation about z -axis: $R_z(+\theta)$

Spring 2013 Utah School of Computing 66

Then rotate about x by $-\beta$: $R_x(-\beta)$

Rotate again in the (y-z)-plane

Spring 2013 Utah School of Computing 7

Now, $+\theta$ Rotation about z by $-\alpha$: $R_z(-\alpha)$

Now to original position of a

Spring 2013 Utah School of Computing 8

We Effected $+\theta$ rotation about Arbitrary axis a : $R_a(+\theta)$

Spring 2013 Utah School of Computing 9

We Effected $+\theta$ rotation about Arbitrary axis a : $R_a(+\theta)$

↓

$$R_a(+\theta) = R_z(-\alpha) R_x(-\beta) R_z(+\theta) \times R_x(+\beta) R_z(+\alpha)$$

Spring 2013 Utah School of Computing 10

Rotation about Arbitrary Axis

- Rotation about a -axis effected by (nonunique) composition of 5 elementary rotations
- We show arbitrary rotation as succession of 5 rotations about principal axes

Spring 2013 Utah School of Computing 11

In matrix terms, $R_a(+\theta) =$

$$R_a(+\theta) = \begin{matrix} R_z(-\alpha) & R_x(-\beta) \\ \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\beta) & -\sin(-\beta) & 0 \\ 0 & \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \times \begin{matrix} R_z(+\theta) & R_x(+\beta) & R_z(+\alpha) \\ \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & 0 \\ 0 & \sin \beta & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Spring 2013 Utah School of Computing 12

Similarly, $R_a^{-1}(+\theta) = R_a(-\theta)$, so

$$R_a(-\theta) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\beta) & -\sin(-\beta) & 0 \\ 0 & \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \underbrace{\begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_z(-\theta)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & 0 \\ 0 & \sin \beta & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 13

Recall, $[AB]^t = B^t A^t$

Consequently, for $A = R^t M R$,

$$A^t = R^t M^t R \quad \text{because,}$$

$$\begin{aligned} [R^t(MR)]^t &= [M \cdot R]^t [R^t]^t \\ &= R^t M^t R \end{aligned}$$

Spring 2013 Utah School of Computing 14

It follows directly that,

$$\left[R^t S^t M S R \right]^t = R^t S^t M^t S R$$

Spring 2013 Utah School of Computing 15

$\therefore R_a^{-1}(+\theta) = R_a^t(\theta)$

$$R_a(-\theta) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\beta) & -\sin(-\beta) & 0 \\ 0 & \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \underbrace{\begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_z^t(\theta)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & 0 \\ 0 & \sin \beta & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 16

Similarly, $R_a^{-1}(+\theta) = R_a(-\theta)$, so

$$R_a(-\theta) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\beta) & -\sin(-\beta) & 0 \\ 0 & \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \underbrace{\begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_z(-\theta)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & 0 \\ 0 & \sin \beta & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 17

In matrix terms, $R_a(+\theta) =$

$$R_a(+\theta) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\beta) & -\sin(-\beta) & 0 \\ 0 & \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_z(+\theta)} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & 0 \\ 0 & \sin \beta & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_x(+\beta)} \underbrace{\begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_z(+\alpha)}$$

Spring 2013 Utah School of Computing 18

Constructively, we have shown,

$$R_a^{-1}(\theta) = R_a^t(\theta)$$

This will be useful later

Spring 2013

Utah School of Computing

19

3D Translation in x

$$T_x(d_x) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+d_x \\ y \\ z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

20

3D Translation in y

$$T_y(d_y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y+d_y \\ z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

21

3D Translation in z

$$T_z(d_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z+d_z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

22

3D Shear in x-direction

$$Sh_x(a) = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+ay \\ y \\ z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

23

3D Shear in x-direction

$$Sh_x(b) = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+bz \\ y \\ z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

24

3D Shears: Clamp a *Principal Plane*, shear in other 2 DoFs

Spring 2013 Utah School of Computing 25

$$Sh_x(a) = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+ay \\ y \\ z \\ 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 26

3D Shear in *x*-direction

$$Sh_x(b) = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+bz \\ y \\ z \\ 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 27

3D Shear in *y*-direction

$$Sh_y(c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y+cz \\ z \\ 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 28

3D Shear in *y*-direction

$$Sh_y(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ dx+y \\ z \\ 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 29

$$Sh_y(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ dx+y \\ z \\ 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 30

3D Shear in y-direction

$$Sh_y(c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y+cz \\ z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

31

3D Shear in z-direction

$$Sh_z(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ ex+z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

32

3D Shear in z

$$Sh_z(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ e & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ ex+z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

33

3D Shear in z

$$Sh_z(f) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ fy+z \\ 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

34

What About Elementary Inverses?

- Scale
- Shear
- Rotation
- Translation

Spring 2013

Utah School of Computing

35

Scale Inverse

$$\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\lambda & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/\lambda & 0 \\ 0 & 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

36

Shear Inverse

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

37

Shear Inverse

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

38

Rotation Inverse

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Spring 2013

Utah School of Computing

39

Rotation Inverse

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \\ = \begin{bmatrix} (\cos^2 \theta + \sin^2 \theta) & (\cos \theta \sin \theta - \cos \theta \sin \theta) \\ (\cos \theta \sin \theta - \cos \theta \sin \theta) & (\sin^2 \theta + \cos^2 \theta) \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

40

Rotation Inverse

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Spring 2013

Utah School of Computing

41

Translation Inverse

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -d_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & (-d_x + d_x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spring 2013

Utah School of Computing

42

Translation Inverse

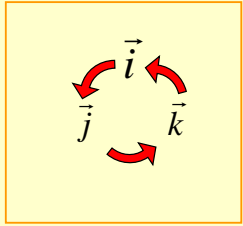
$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -d_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spring 2013 Utah School of Computing 43

Want the RHR to Work

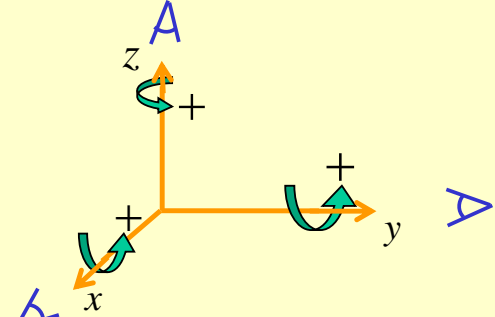
$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j}$$


Spring 2013 Utah School of Computing 44

3D Positive Rotations



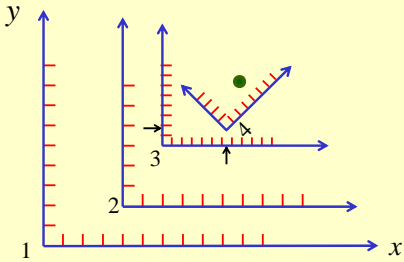
Spring 2013 Utah School of Computing 45

Transf's as Change in Coordinate Sys

- Useful in many situations
- Use most natural coordination system locally
- Tie things together in a global system

Spring 2013 Utah School of Computing 46

Example



Spring 2013 Utah School of Computing 47

Example

$M_{i \leftarrow j}$ is the transformation that takes a point $p^{(j)}$ in coordinate system j and converts it to a point $p^{(i)}$ in coordinate system i

Spring 2013 Utah School of Computing 48

Example

- $p^{(i)} = M_{i \leftarrow j} p^{(j)}$
- $p^{(j)} = M_{j \leftarrow k} p^{(k)}$
- $M_{i \leftarrow k} = M_{i \leftarrow j} M_{j \leftarrow k}$

Spring 2013 Utah School of Computing 49

Example

- $M_{1 \leftarrow 2} = T(4, 2)$
- $M_{2 \leftarrow 3} = S(2, 2) \cdot T(2, 3)$
- $M_{3 \leftarrow 4} = R(-45^\circ) \cdot T(6.7, 1.8)$

Spring 2013 Utah School of Computing 50

Recall the Following

$$(AB)^{-1} = B^{-1} A^{-1}$$

Spring 2013 Utah School of Computing 51

Since $M_{i \leftarrow j}^{-1} = M_{j \leftarrow i}$

- $M_{2 \leftarrow 1} = T(-4, -2)$
- $M_{3 \leftarrow 2} = T(-2, -3) \cdot S\left(\frac{1}{2}, \frac{1}{2}\right)$
- $M_{4 \leftarrow 3} = T(-6.7, -1.8) \cdot R(+45^\circ)$

Spring 2013 Utah School of Computing 52

Example

Spring 2013 Utah School of Computing 53

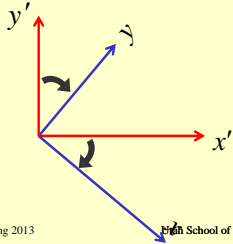
Change of Coordinate System

- Describe the *old* coordinate system in terms of the *new* one.

Spring 2013 Utah School of Computing 54

Change of Coordinate System

Move to the *new* coordinate system and describe the one *old*.



Old is a negative rotation of the *new*.