Bresenham's Midpoint Algorithm

CS5600 Computer Graphics
adapted from
Rich Riesenfeld's slides
Spring 2013

Lecture Set 2

Line Characterizations

- Explicit: $y = mx + B$
- Implicit: $F(x,y) = ax + by + c = 0$
- Constant slope: $\frac{\Delta y}{\Delta x} = k$
- Constant derivative: $f'(x) = k$

Line Characterizations - 2

- Parametric: $P(t) = (1-t)\,P_0 + t\,P_1$

  where, $P(0) = P_0\,;\quad P(1) = P_1$
- Intersection of 2 planes
- Shortest path between 2 points
- *Convex hull* of 2 discrete points

Two Line Equations

- Explicit: $y = mx + B$
- Implicit: $F(x,y) = ax + by + c = 0$

  Define: $dy = y_1 - y_0$

  $dx = x_1 - x_0$

  Hence, $y = \left(\frac{dy}{dx}\right)x + B$

From previous

We have, $y = \left(\frac{dy}{dx}\right)x + B$

Hence, $\frac{dy}{dx}x - y + B = 0$

Relating Explicit to Implicit Eq's

Recall, $\frac{dy}{dx}x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$\therefore\ F(x,y) = (dy)x + (-dx)y + (dx)B = 0$

where, $a = (dy);\quad b = -(dx);\quad c = B(dx)$

Discrete Lines

- Lines vs. Line Segments
- What is a discrete line segment?
  - This is a relatively recent problem
  - How to generate a discrete line?

"Good" Discrete Line

- No gaps in adjacent pixels
- Pixels close to ideal line
- Consistent choices; same pixels in same situations
- Smooth looking
- Even brightness in all orientations
- Same line for $P_0 P_1$ as for $P_1 P_0$
- Double pixels stacked up?

How to Draw a Line?

1. Compute slope
2. Start at on point (x_o, y_o)
3. Increment Δx and draw
   - How to figure this out?

Derive from Line Equation

$Y = mX + b$
$Y_i = mX_i + b$
$X_{i+1} = X_i + \Delta X$
$Y_{i+1} = mX_{i+1} + b$
$Y_{i+1} = m(X_i+\Delta X) + b$
$= mX_i + m\Delta X + b$
$= m\Delta X + mX_i + b$
$= m\Delta X + Y_i$

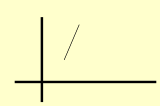
Derive from Line Equation

$Y = mX + b$
$Y_i = mX_i + b$
$X_{i+1} = X_i + \Delta X$
$Y_{i+1} = mX_{i+1} + b$
$Y_{i+1} = m(X_i+\Delta X) + b$
$= mX_i + m\Delta X + b$
$= m\Delta X + mX_i + b$
$= m\Delta X + Y_i$

$m = \Delta y / \Delta x$
$= (y_1 - y_0) / (x_1 - x_0)$
$\Delta y = \Delta x \, (y_1 - y_0) / (x_1 - x_0)$
$Y_{i+1} = Y_i + \Delta Y$
$= Y_i + \Delta x \, (y_1 - y_0) / (x_1 - x_0)$
$= Y_i + \Delta X(m)$

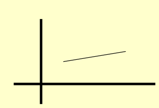
What about slope?

If $|y_1 - y_0| > |x_1 - x_0|$
increment Y



else

increment X



Restricted Form

- Line segment in *first* octant with

  $0 < m < 1$

- After we derive this, we'll look at the other cases (other octants)

Incremental Fn Eval

- Recall $f(x_{i+1}) = f(x_i) + \Delta(x_i)$
- Characteristics
  - Fast
  - Cumulative Error
- Need to define $f(x_o)$

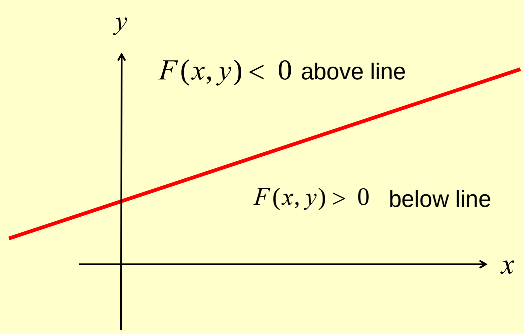
Investigate Sign of F

Verify that

$$F(x,y) = \begin{cases} + & \text{below line} \\ 0 & \text{on line} \\ - & \text{above line} \end{cases}$$

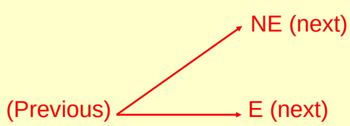
Look at extreme values of y

The Picture



Key to Bresenham Algorithm

"Reasonable assumptions" have reduced the problem to making a binary choice at each pixel:

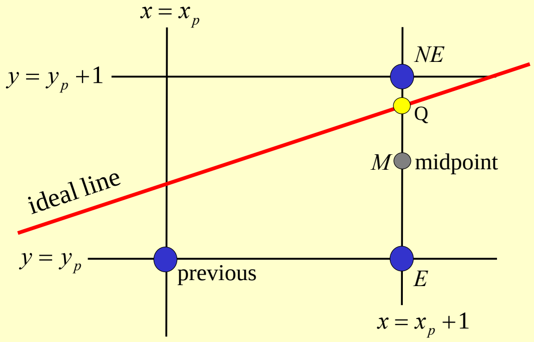


Decision Variable d (logical)

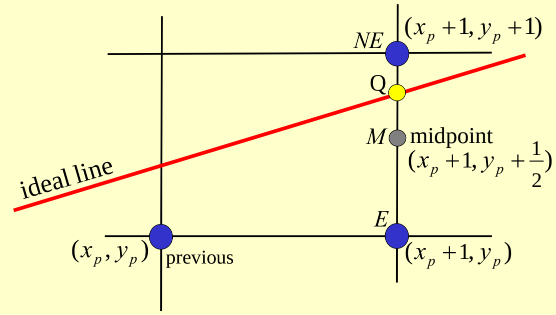
Define a logical *decision* variable d

- linear in form
- incrementally updated (with addition)
- tells us whether to go *E* or *NE*

The Picture



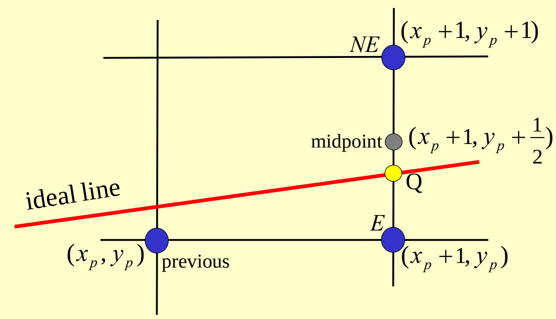
The Picture (again)



Observe the relationships

- Suppose Q is above M, as before.
- Then $F(M) > 0$, *M* is below the line
- So, $F(M) > 0$ means line is above M,
- Need to move *NE, increase y* value

The Picture (again)



Observe the relationships

- Suppose Q is below M, as before.
- Then $F(M) < 0$, implies M is *above* the line
- So, $F(M) < 0$, means line is below M,
- Need to move to *E; don't increase* y

M = Midpoint $= (x_p + 1, y_p + \frac{1}{2})$

- Want to evaluate at M
- Will use an incremental *decision* variable d

$$d = F(x_p + 1, y_p + \frac{1}{2})$$

- Let, $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

How will d be used?

Let, $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

Therefore,

$$d = \begin{cases} > 0 & \Rightarrow NE \quad \text{(midpoint below ideal line)} \\ < 0 & \Rightarrow E \quad \text{(midpoint above ideal line)} \\ = 0 & \Rightarrow E \quad \text{(arbitrary)} \end{cases}$$

Case E: Suppose E is chosen

- Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$
- $E \Rightarrow: \quad x \leftarrow x + 1; \quad y \leftarrow y,$
- $\therefore .... d_{new} = F(x_p + 2, y_p + \frac{1}{2})$

$$= a(x_p + 2) + b(y_p + \frac{1}{2}) + c$$

Case E: Suppose E is chosen

$$d_{new} - d_{old} = \left(a(x_p + 2) + b(y_p + \frac{1}{2}) + c \right)$$

$$-\left(a(x_p + 1) + b(y_p + \frac{1}{2}) + c \right)$$

$$\boxed{d_{new} = d_{old} + a}$$

Review of Explicit to Implicit

Recall, $\frac{dy}{dx} x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$$\therefore \quad F(x, y) = (dy)x + (-dx)y + (dx)B = 0$$

where, $a = (dy); \quad b = -(dx); \quad c = B(dx)$

Case E: $d_{new} = d_{old} + a$

$\Delta_E \equiv$ increment we add if E is chosen.
So, $\Delta_E = a$. But remember that
$a = dy$ (from line equations).
Hence, $F(M)$ is not evaluated explicitly.
We simply add $\boxed{\Delta_E = a}$ to update d for E

Case NE: Suppose *NE* chosen

Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

and, $NE \Rightarrow: \quad x \leftarrow x + 1; \quad y \leftarrow y + 1,$

$\therefore \quad d_{new} = F(x_p + 2, y_p + \frac{3}{2})$

$$= a(x_p + 2) + b(y_p + \frac{3}{2}) + c$$

Case NE: Suppose *NE*

$$d_{new} - d_{old} =$$

$$= \left(a(x_p + 2) + b(y_p + \frac{3}{2}) + c \right)$$

$$- \left(a(x_p + 1) + b(y_p + \frac{1}{2}) + c \right)$$

$$\boxed{d_{new} = d_{old} + a + b}$$

Case NE: $d_{new} = d_{old} + a + b$

$\Delta_{NE} \equiv$ increment that we add if NE is chosen.
So, $\Delta_{NE} = a + b$. But remember that
$a = dy$, and $b = -dx$ (from line equations).
Hence, $F(M)$ is not evaluated explicitly.
We simply add $\Delta_{NE} = a + b$ to update d for NE

Case NE: $d_{new} = d_{old} + a + b$

$\Delta_{NE} = a + b$, where $a = dy$, and $b = -dx$

means, we simply add $\Delta_{NE} = a + b$, i.e.,

$\boxed{\Delta_{NE} = dy - dx}$ to update d for NE.

Summary

- At each step of the procedure, we must choose between moving *E* or *NE* based on the sign of the decision variable d
- Then update according to

$$d \leftarrow \begin{cases} d + \Delta_E, & \text{where } \Delta_E = dy, \text{or} \\ d + \Delta_{NE}, & \text{where} \Delta_{NE} = dy - dx \end{cases}$$

What is initial value of d ?

- First point is (x_0, y_0)
- First midpoint is $(x_0 + 1, y_0 + \frac{1}{2})$
- What is initial midpoint value?

$$d(x_0 + 1, y_0 + \frac{1}{2}) = F(x_0 + 1, y_0 + \frac{1}{2})$$

What is initial value of d ?

$$F(x_0 + 1, y_0 + \frac{1}{2}) = a(x_0 + 1) + b(y_0 + \frac{1}{2}) + c$$

$$= (ax_0 + by_0 + c) + \left(a + \frac{b}{2} \right)$$

$$= F(x_0, y_0) \quad + \quad \left(a + \frac{b}{2} \right)$$

What is initial value of d ?

Note, $F(x_0, y_0) = 0$, since (x_0, y_0) is on line.

Hence, $F(x_0 + 1, y_0 + \frac{1}{2}) = 0 + a + \frac{b}{2}$

$$= (dy) - \left(\frac{dx}{2}\right)$$

What Does Factor of 2 x Do ?

- Has the same 0-set
  $2F(x, y) = 2(ax + by + c) = 0$
- Changes the slope of the plane
- Rotates plane about the 0-set line
- Gets rid of the denominator

What is initial value of d ?

Note, we can clear denominator and not change line,

$$2F(x_0 + 1, y_0 + \frac{1}{2}) = 2(dy) - dx$$

What is initial value of d ?

$2F(x, y) = 2(ax + by + c) = 0$

So, first value of

$$d = 2(dy) - (dx)$$

More Summary

- Initial value $2(dy) - (dx)$
- Case E: $d \leftarrow d + \Delta_E$, where $\Delta_E = 2(dy)$
- Case NE: $d \leftarrow d + \Delta_{NE}$,
  where $\Delta_{NE} = 2\{(dy) - (dx)\}$
- Note, all deltas are constants

More Summary

Choose $\begin{cases} E & \text{if } d \le 0 \\ NE & \text{otherwise} \end{cases}$

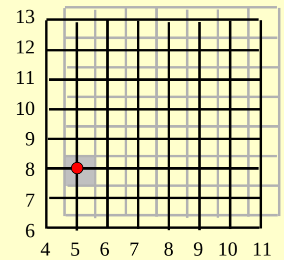
Example

- Line end points:

  $(x_0, y_0) = (5,8); \quad (x_1, y_1) = (9,11)$

- Deltas: $dx = 4; \; dy = 3$

Graph



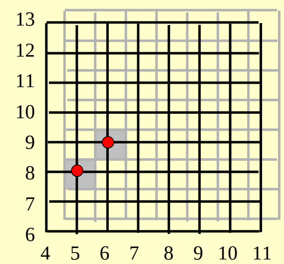
Example ($dx = 4; \; dy = 3$)

- Initial value of

$$d(5,8) = 2(dy) - (dx)$$
$$= 6 - 4 = 2 > 0$$
$$d = 2 \;\Rightarrow\; NE$$

$\text{d} = 2(dy) - (dx)$ $\quad \begin{cases} E & \text{if } d \le 0 \\ NE & \text{otherwise} \end{cases}$

Graph



Example ($dx = 4; \; dy = 3$)

- Update value of d

  $d \leftarrow d + \Delta_E$, where $\Delta_E = 2(dy)$

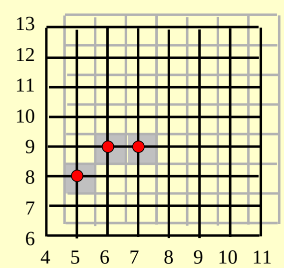
  $d \leftarrow d + \Delta_{NE}$, where $\Delta_{NE} = 2\{(dy) - (dx)\}$

- Last move was *NE*, so

$$\Delta_{NE} = 2(dy - dx)$$
$$= 2(3 - 4) = -2$$
$$d = 2 - 2 = 0 \;\Rightarrow\; E$$

$\begin{cases} E & \text{if } d \le 0 \\ NE & \text{otherwise} \end{cases}$

Graph



Example ($dx = 4;\ dy = 3$)

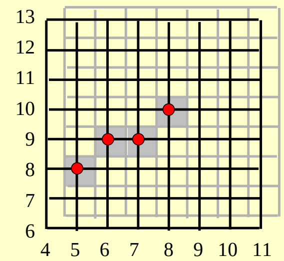
$d \leftarrow d + \Delta_E$, where $\Delta_E = 2(dy)$

- Previous move was *E* $\quad d \leftarrow d + \Delta_{NE}$, where $\Delta_{NE} = 2\{(dy) - (dx)\}$

$$\Delta_E = 2(dy)$$
$$= 2(3) = 6$$
$$d = 0 + 6 > 0 \quad \Rightarrow \quad NE$$

$$\begin{cases} E & \text{if } d \le 0 \\ NE & \text{otherwise} \end{cases}$$

Graph



Example ($dx = 4;\ dy = 3$)

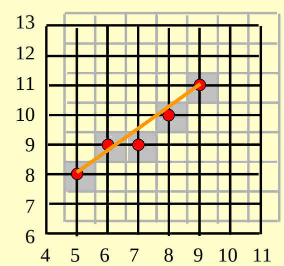
$d \leftarrow d + \Delta_E$, where $\Delta_E = 2(dy)$

- Previous move was *NE* $\quad d \leftarrow d + \Delta_{NE}$, where $\Delta_{NE} = 2\{(dy) - (dx)\}$

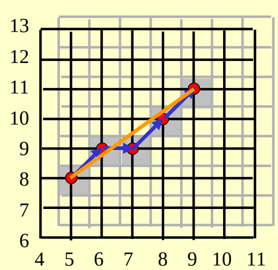
$$\Delta_{NE} = 2(dy - dx)$$
$$= 2(3 - 4) = -2$$
$$d = 6 - 2 = 4 \quad \Rightarrow \quad NE$$

$$\begin{cases} E & \text{if } d \le 0 \\ NE & \text{otherwise} \end{cases}$$

Graph



Graph



Meeting Bresenham Criteria

Case 0: $m = 0; \quad m = 1 \quad \Rightarrow$ trivial cases

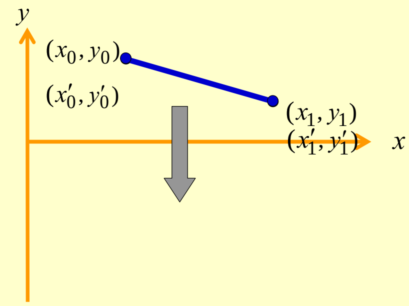
Case 1: $0 > m > -1 \quad \Rightarrow$ flip about x-axis

Case 2: $m > 1 \quad \Rightarrow$ flip about $x = y$

Case 0: Trivial Situations

- $m = 0 \quad \Rightarrow \quad$ horizontal line
- $m = 1 \quad \Rightarrow \quad$ line $y = x$
- Do not need Bresenham

Case 2: Flip about *x-axis*



Case 2: Flip about *x-axis*

- Suppose, $\quad 0 > m > -1$,
- Flip about x-axis $(y' = -y)$:

$$(x'_0, y'_0) = (x_0, -y_0);$$
$$(x'_1, y'_1) = (x_1, -y_1)$$

How do slopes relate?

$$m = \frac{y_1 - y_0}{x_1 - x_0};$$
$$m' = \frac{y'_1 - y'_0}{x_1 - x_0}$$

by definition

Since $y'_i = -y_i$, $\quad m' = \frac{-y_1 - (-y_0)}{x_1 - x_0}$

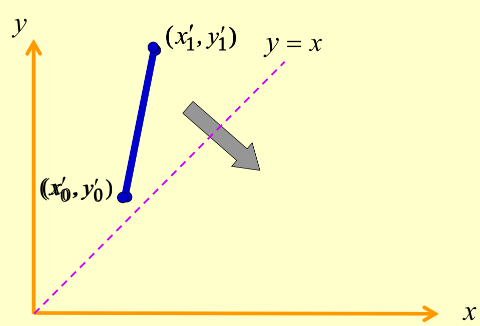
How do slopes relate?

i.e., $\quad m' = -\frac{(y_1 - y_0)}{x_1 - x_0}$

$$m' = -m$$

$\therefore \quad 0 > m > -1 \quad \Rightarrow \quad 0 < m' < 1$

Case 3: Flip about line $y=x$



Case 3: Flip about line $y=x$

$$y = mx + B,$$

swap $x \leftrightarrow y$ and prime them ,

$$x' = my' + B,$$

$$my' = x' - B$$

Case 3: $m' = ?$

$$y' = \left(\frac{1}{m}\right)x' - B,$$

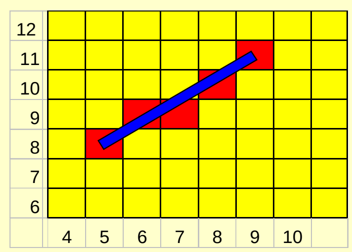
$$\therefore \quad m' = \left(\frac{1}{m}\right) \quad \text{and,}$$

$$m > 1 \quad \Rightarrow \quad 0 < m' < 1$$

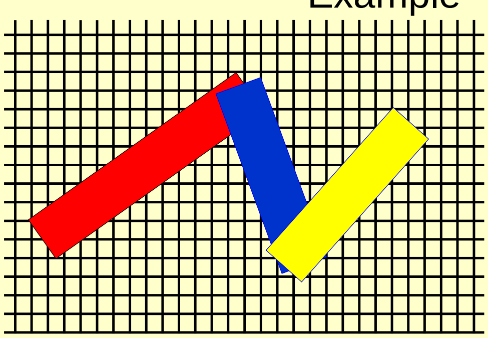
More Raster Line Issues

- Fat lines with multiple pixel width
- Symmetric lines
- How should end pt geometry look?
- Generating curves, e.g., circles, etc.
- Jaggies, staircase effect, aliasing...

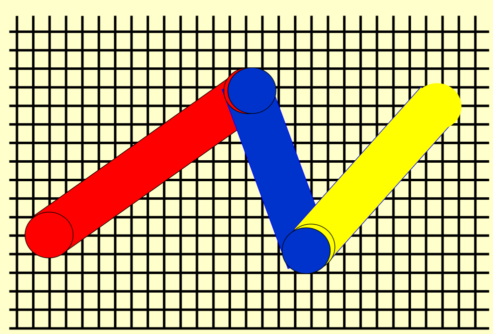
Pixel Space



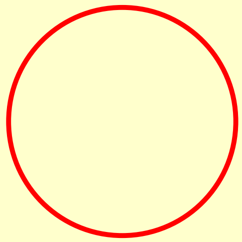
Example



Example



Bresenham Circles



The End
Bresenham's Algorithm

Lecture Set 2