Quiz

What type is inferred for `?` in the following expression?

```
{with {f : (? -> ?) {fun {x : ?} x}}
  {f 10}}
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Answer: *num*

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```

Answer: $(\mathit{num} \rightarrow \mathit{num})$

Quiz

What type is inferred for `?` in the following expression?

```
{with {f : (? -> ?) {fun {x : ?} x}}
  {if0 ...
       {f 10}
       {{f {fun {x : num} x}} 8}}}
```

Quiz

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{with {f : (? -> ?) {fun {x : ?} x}}
  {if0 ...
       {f 10}
       {{f {fun {x : num} x}} 8}}}
```

Answer: None; no single τ works — but it's a perfectly good program for any ... of type `num`

Polymorphism

We'd like a way to write a type that the caller chooses:

```
{with {f : ?
          [tyfun [alpha]
                 {fun {x : alpha} x}]}
  {if0 …
       {[@ f num] 10}
       {{[@ f (num -> num)] {fun {x : num} x}} 8}}}
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       {[@ f num] 10}
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```

This `f` is ***polymorphic***

- The `tyfun` form parameterizes over a type
- The `@` form picks a type

Polymorphic Types

What is the type of this expression?

```
[tyfun [alpha]
       {fun {x : alpha} x}]
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[tyfun [alpha]
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It should be something like (**`alpha`** $\rightarrow$ **`alpha`**), but it needs a specific type before it can be used as a function

Polymorphic Types

What is the type of this expression?

```
[tyfun [alpha]
       [tyfun [beta]
              {fun {x : alpha} x}]]
```

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It should be something like (`alpha` → `alpha`), but picking `alpha` gives something that still needs another type

Polymorphic Types

What is the type of this expression?

```
[tyfun [alpha]
       [tyfun [beta]
              {fun {x : alpha} x}]]
```

It should be something like (**alpha** → **alpha**), but picking **alpha** gives something that still needs another type

New type form: ∀**<tyid>**.**<TE>**

∀**alpha**.(**alpha** → **alpha**)

∀**alpha**.∀**beta**.(**alpha** → **alpha**)

TPFAE Grammar

```
<TPFAE>  ::=  <num>
           |  {+ <TPFAE> <TPFAE>}
           |  {- <TPFAE> <TPFAE>}
           |  <id>
           |  {fun {<id> : <TE>} <TPFAE>}
           |  {<TPFAE> <TPFAE>}
           |  {if0 <TPFAE> <TPFAE> <TPFAE>}
           |  [tyfun [<tyid>] <TPFAE>]          NEW
           |  [@ <TPFAE> <TE>]                  NEW

<TE>     ::=  num
           |  (<TE> -> <TE>)
           |  (forall <tyid> <TE>)              NEW
           |  <tyid>                            NEW
```

TPFAE Type Checking

$$\frac{\Gamma[\texttt{<tyid>}] \vdash \texttt{e} : \tau}{\Gamma \vdash \texttt{[tyfun [<tyid>] e]} : \forall\texttt{<tyid>}.\tau}$$

$$\frac{\Gamma \vdash \tau_0 \qquad \Gamma \vdash \texttt{e} : \forall\texttt{<tyid>}.\tau_1}{\Gamma \vdash \texttt{[@ e } \tau_0\texttt{]} : \tau_1[\texttt{<tyid>}\leftarrow\tau_0]}$$

$$[\ldots\texttt{<tyid>}\ldots] \vdash \texttt{<tyid>}$$

$$\frac{\Gamma[\texttt{<tyid>}] \vdash \tau}{\Gamma \vdash \forall\texttt{<tyid>}.\tau}$$

Polymorphism and Type Definitions

If we mix **`tyfun`** with **`withtype`**, then we can write

```
{with {f : (forall alpha (alpha -> num))
         [tyfun [alpha]
                {fun {v : alpha}
                     {withtype {list {empty num}
                                     {cons (alpha * list)}}
                               {rec {len : (list -> num)
                                         {fun {l : list}
                                              {cases list l
                                                {empty {n} 0}
                                                {cons {fxr}
                                                      {+ 1 {len {snd fxr}}}}}}}
                                    {len {cons {pair v
                                                     {cons {pair v
                                                                 {empty 0}}}}}}}}}]}
  {+ {[@ f num] 10}
     {[@ f (num -> num)] {fun {x : num} x}}}}
```

This is a kind of polymorphic list definition

Problem: everything must be under a **`tyfun`**

Polymorphism and Type Definitions

Solution: build `tyfun`-like abstraction into `withtype`

```
{withtype {{alpha list} {empty num}
           {cons (alpha * {alpha list})}}
          {rec {len : (forall alpha ({alpha list} -> num))
                  [tyfun [alpha]
                         {fun {l : {alpha list}}
                              {cases {alpha list} l
                                {empty {n} 0}
                                {cons {fxr}
                                      {+ 1 {len {snd fxr}}}}}}]}
               {+ {[@ len num] {[@ cons num] {pair 1 {[@ empty num] 0}}}}
                  {[@ len (num -> num)] {[@ empty (num -> num)] 0}}}}}
```

Polymorphism and Inference

```
{with {f : (forall alpha (alpha -> alpha))
           [tyfun [alpha]
                  {fun {x : alpha}
                       x}]}
   {[@ f (num -> num)] {fun {y : num} y}}}
```

The type application `[@ f (num -> num)]` is obvious, since we can get the type of `{fun {y : num} y}`

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With polymorphism, type inference is usually combined with type-application inference:

```
{with {f : (forall alpha (alpha -> alpha))
          [tyfun [alpha]
                 {fun {x : alpha}
                      x}]}
  {f {fun {y : num} y}}}
```

Polymorphism and Inference

```
{with {f : ?
          {fun {x : ?}
                 x}}
  {f {fun {y : num} {f 10}}}}
```

How about inferring a **`tyfun`** around the value of **`f`**?

Polymorphism and Inference

```
{with {f : ?
          {fun {x : ?}
               x}}
  {f {fun {y : num} {f 10}}}}
```

How about inferring a `tyfun` around the value of `f`?

Yes, with some caveats...

Polymorphism and Inference

Does the following expression have a type?

```
{fun {x : ?} {x x}}
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Yes, if we infer `forall` types and type applications:

```
{fun {x : (forall alpha (alpha -> alpha))}
     {[@ x (num -> num)] [@ x num]}}
```

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Does the following expression have a type?

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Yes, if we infer `forall` types and type applications:

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{fun {x : (forall alpha (alpha -> alpha))}
     {[@ x (num -> num)] [@ x num]}}
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Inferring types like this is arbitrarily difficult (i.e., undecidable), so type systems generally don't

Let-Based Polymorphism

Inference constraint: only infer a polymorphic type (and insert **`tyfun`**) for ther right-hand side of a **`with`** or **`rec`** binding

- This works:

```
{with {f : ?
           {fun {x : ?}
                x}}
  {f {fun {y : num} {f 10}}}}
```

- This doesn't:

```
{fun {x : ?} {x x}}
```

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- This doesn't:

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{fun {x : ?} {x x}}
```

Note: makes `with` a core form

Implementation: check right-hand side, add a `forall` and `tyfun` for each unconstrained *new* type variable

Polymorphism and Inference and Type Definitions

All three together make a practical programming system:

```
{withtype {{alpha list} {empty num}
            {cons (alpha * {alpha list})}}
           {rec {len : ?
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                             {empty {n} 0}
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                {+ {len {cons {pair 1 {empty 0}}}}
                   {len {cons {pair {fun {x : num} x} {empty 0}}}}}}}
```

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               {+ {len {cons {pair 1 {empty 0}}}}
                  {len {cons {pair {fun {x : num} x} {empty 0}}}}}}}
```

Caml example:

```
type 'a tree = Leaf of 'a
             | Fork of 'a tree * 'a tree
```

Polymorphism and Values

A ***polymorphic function*** is not quite a function:

- A **function** is applied to a value to get a new value
- A **polymorphic function** is applied to a type to get a function

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What happens if you write the following?

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{with {f : ? {fun {g : ?}
                {fun {v : ?}
                     {g v}}}}
  {with {g : ? {fun {x : ?} x}}
    {{f g} 10}}}
```

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                       {g v}}}}
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```

A type application must be used at the function call, not in `f`:

```
{{[@ [@ f num] num] 10} [@ g num]}
```

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    {{f 10} g}}}
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                  {fun {g : (forall alpha (alpha -> alpha))}
                       {g v}}}}
  {with {g : ? {fun {x : ?} x}}
    {{f 10} g}}}
```

One type application must be used inside `f`:

```
[tyfun {beta} {fun {v : beta}
                   {fun {g : (forall alpha (alpha -> alpha))}
                        {[@ g beta] v}}}]
```

Polymorphism and Values

An argument that is a polymorphic value can be used in multiple ways:

```
{fun {g : (forall alpha (alpha -> alpha))}
     {if {g false}
         {g 0}
         {g 1}}}
```

Polymorphism and Values

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```
{fun {g : (forall alpha (alpha -> alpha))}
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```

but due to inference constraints,

```
{fun {g : ?}
     {if {g false}
         {g 0}
         {g 1}}}
```

would be rejected!

Polymorphism and Values

ML prohibits polymorphic values, so that

```
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is not allowed

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```
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         {g 1}}}
```

is not allowed

- Consistent with inference
- Every `forall` appears at the beginning of a type, so

  ```
  (forall alpha (forall beta (alpha -> beta)))
  ```

  can be abbreviated

  ```
  (alpha -> beta)
  ```

  without loss of information