CS3130: Probability and Statistics for Engineers

Practice Final Solutions

1. (a) Probability of being male, given that you have autism spectrum disorder. $P(M|A) = 0.8$ (given in problem).

   (b) Probability of having autism spectrum disorder, given that you are a male. Using Bayes' Rule:

$$P(A|M) = \frac{P(M|A)P(A)}{P(M)} = \frac{0.8 \times 0.006}{0.5} = 0.0096$$

   (c)

$$\begin{aligned} P(M^c \cap A) &= P(A - M) && \text{definition of set minus} \\ &= P(A) - P(A \cap M) && \text{difference rule} \\ &= P(A) - P(M|A)P(A) && \text{multiplication rule} \\ &= 0.006 - 0.8 \times 0.006 \\ &= 0.0012 \end{aligned}$$

2. (a) Probability of success on first attepmt: $P(X = 1) = p$. So, probability that it *isn't* on the first attempt is: $P(X > 1) = 1 - P(X = 1) = 1 - p$.

   (b) Using the pmf for the Geometric distribution and the formula given in the problem, this is:

$$P(X \le n) = \sum_{i=1}^{n} P(X = i) = \sum_{i=1}^{n} (1-p)^{i-1} p = 1 - (1-p)^n$$

   (c) This is just the complement of (b), so we get

$$P(X > n) = 1 - P(X \le n) = 1 - (1 - (1-p)^n) = (1-p)^n$$

   (d) Here we apply the standard formula for conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, with $A = \{X = k + n\}$ and $B = \{X > n\}$. Notice that if $X = k + n$ is true, then $X > n$ is also true, so $A \cap B = A$ in this case. We get:

$$P(X = k{+}n \,|\, X > n) = \frac{P(X = k+n)}{P(X > n)} = \frac{(1-p)^{k+n-1} p}{(1-p)^n} = (1{-}p)^{k-1} p = P(X = k)$$

3. (a) The cdf must be equal to one at the end of its range (at $x = \pi$). That is,

$$1 = F_X(\pi) = k(\pi - \sin(\pi)) = k\pi$$

   Solving for k gives us $k = \frac{1}{\pi}$.

   (b) For x in the range $0 \le x \le \pi$, we get

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \frac{d}{dx} \frac{1}{\pi}(x + \sin(x)) \\ &= \frac{1}{\pi}(1 + \cos(x)) \end{aligned}$$

   For all other values of x, $f_X(x) = 0$.

(c)

$$\begin{aligned} P\left(X \le \frac{\pi}{2}\right) &= F_X\left(\frac{\pi}{2}\right) \\ &= \frac{1}{\pi}\left(\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)\right) \\ &= \frac{1}{2} + \frac{1}{\pi} \end{aligned}$$

(d)

$$E[X] = \frac{1}{\pi}\int_0^{\pi} x(1+\cos(x))dx$$

4. (a) Because X and Y are uncorrelated $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 2\sigma^2$.
   (b) Note from the definition of correlation that

$$\mathrm{Cov}(X,Y) = \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}\rho(X,Y) = -\sigma^2.$$

   This gives

$$\begin{aligned} \mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) \\ &= 2\sigma^2 - 2\sigma^2 \\ &= 0 \end{aligned}$$

   (c) For part (a) it is “not enough information”. Uncorrelated does not imply independent. For part (b) they are “dependent”. Correlated (either positive or negative) variables are always dependent.

5. First, for a 99% confidence interval, we are using the level $\alpha = 0.01$. This means we want to use $z_{\alpha/2} = z_{0.005} = 2.58$. Now, the confidence interval is (l_n, u_n):

$$l_n = \bar{x}_n - z_{0.005}\frac{\sigma}{\sqrt{n}} = 0.305 - 2.58 \times \frac{0.25}{10} = 0.3695$$

$$u_n = \bar{x}_n + z_{0.005}\frac{\sigma}{\sqrt{n}} = 0.305 + 2.58 \times \frac{0.25}{10} = 0.2405$$

6. (a) Assuming the monthly sales come from a Gaussian distribution, $N(\mu, \sigma^2)$.

$$\begin{aligned} H_0 &: \mu = 800 \qquad \text{(average monthly sales are equal to 800 units)} \\ H_1 &: \mu > 800 \qquad \text{(average monthly sales are greater than 800)} \end{aligned}$$

   (b) You should use the critical value $F^{-1}(0.95) = 2.35$ because you are doing a one-sided test to the right.
   (c) Using the T statistic formula:

$$t = \frac{\bar{x}_n - \mu_0}{s_n/\sqrt{n}} = \frac{805 - 800}{10/\sqrt{4}} = \frac{5}{5} = 1$$

   (d) No, the t statistic is not greater than our critical value.