Summary of Methods

① MOM:

Begin with integral eq^n

$$\int_V \underbrace{u(\bar{r}')}_{\text{unknown source}} g(\bar{r},\bar{r}')\,dv' = f(\bar{r})$$

Divide region into N sections &
Write unknown as sum of basis f^{n}s

$$u(\bar{r}') \cong \sum_{i=1}^{N} \underbrace{b_i(\bar{r}')}_{\text{basis } f^n} \underbrace{A_i}_{\text{unknown}}$$

Now, if you could find A_is, you would know $u(r')$.

Substitute into integral eq^n

$$\int_V \left(\sum_{i=1}^{N} b_i(\bar{r}')\,A_i\right) g(\bar{r},\bar{r}')\,dv' \cong f(\bar{r})$$

Exchange $\int$ & Σ

$$\sum_{i=1}^{N} A_i \underbrace{\int_V b_i(\bar{r}')\,g(\bar{r},\bar{r}')\,dv'}_{g_i(\bar{r})} \cong f(r)$$

$$\sum_{i=1}^{N} A_i\, g_i(\bar{r}) \cong f(r) \qquad \text{(1 Eq}^n\text{ N unknowns)}$$

The residual of this method is:

$$\sum_{i=1}^{N} A_i\, g_i(\bar{r}) - f(\bar{r}) = R \sim 0$$

Weight the residual over each section &
"add them up" (integrate)

$$\int_v w_m(\bar{r}) \cdot R \, dv = 0$$

$$\int_v w_m(\bar{r}) \cdot \left(\sum_{i=1}^{N} A_i\, g_i(\bar{r}) \right) dv = \int_v w_m(\bar{r}) \cdot f(\bar{r}) \, dv$$

Again switch $\int$ & Σ

$$\sum_{i=1}^{N} A_i \int_v w_m(\bar{r}) \cdot g_i(\bar{r}) \, dv = \int_v w_m(\bar{r}) \cdot f(\bar{r}) \, dv$$

Write in terms of inner product $\langle w, g \rangle = \int_v w \cdot g \, dv$

$$\sum_{i=1}^{N} A_i \langle w_m(\bar{r}), g_i(\bar{r}) \rangle = \langle w_m(\bar{r}), f(\bar{r}) \rangle$$

Write as matrix eq$^{\underline{n}}$:

$$\begin{bmatrix} \langle w_1, g_1 \rangle & \langle w_1, g_2 \rangle & \cdots & \\ \langle w_2, g_1 & \langle w_2, g_2 \rangle & & \\ & & & \\ & & & \langle w_N, g_N \rangle \end{bmatrix} = \begin{bmatrix} \langle w_1, f \rangle \\ \\ \\ \langle w_N, f \rangle \end{bmatrix}$$

② FEM

Begin by assuming the form of the unknown function over one element

$$U_e(\bar{r}) = a + bx + cy \qquad \text{for instance}$$

Write this eqn at each node, and solve for elemental shape functions

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = f^{\underline{n}} \text{ element no. e locat. \& unknowns}$$

$$U_e(\bar{r}) = [1 \quad x \quad y] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= [\alpha_1 \quad \alpha_2 \quad \alpha_3] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$U_e(\bar{r}) = \sum_{i=1}^{N_e} \alpha_i U_i$$

unknowns (Like Ais)

shape $f^{\underline{n}}$s (like bis)

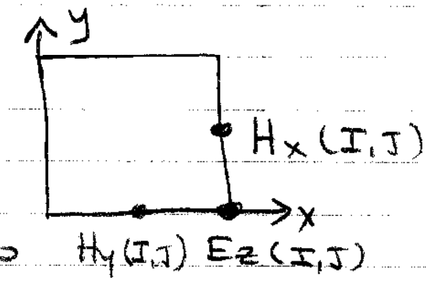
2D Maxwell's Eq^{n}s $\quad (\partial/\partial z = 0,\ H_z = E_x = E_y = 0)$

$$-\mu \frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y}$$

$$-\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$\left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$

FDTD Equations



$$\mu \frac{\partial H_x}{\partial t} + \frac{E_z(I,J+1) - E_z(I,J)}{\Delta y} = 0$$

$$\mu \frac{\partial H_y}{\partial t} + \frac{E_z(I,J) - E_z(I-1,J)}{\Delta x} = 0$$

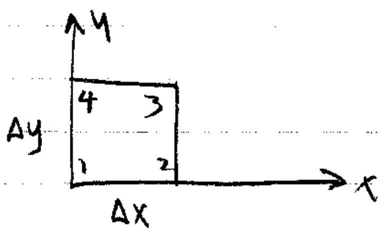
$$\left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) E_z(I,J) + \frac{H_x(I,J) - H_x(I,J-1)}{\Delta y} - \frac{H_y(I+1,J) - H_y(I,J)}{\Delta x} = 0$$

Bring in the weighting fns...
Weight the residuals in each element, and make sum of weighted residuals $=0$

$$\sum_{\#elements} \int_{Se} \overline{W} \left[\mu \frac{\partial}{\partial t} \overline{N_{Hx}}\,\overline{H_{xi}} + \frac{\partial}{\partial y} \overline{N_{Ez}}\,\overline{E_{zi}} \right] dSe = 0$$

Choose a square element

his ssumes E_z, H_x, H_y are all located at each node.



$$N_1 = (1 - x/\Delta x)(1 - y/\Delta y)$$
$$N_2 = (x/\Delta x)(1 - y/\Delta y)$$
$$N_3 = (x/\Delta x)(y/\Delta y)$$
$$N_4 = (1 - x/\Delta x)(y/\Delta y)$$

$$\overline{N} = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

$$\int_{Se} \overline{N}\, dse = \int_{x=0}^{hx} \int_{y=0}^{hy} \left[\left(1 - \frac{x}{\Delta x}\right)\left(1 - \frac{y}{\Delta y}\right) \quad \left(\frac{x}{\Delta x}\right)\left(1 - \frac{y}{\Delta y}\right) \quad \cdots \right] dx\, dy$$

$$= \left[\left(x - \frac{x^2}{2\Delta x}\right)\left(y - \frac{y^2}{2\Delta y}\right) \cdots \right] \Bigg|_{x=0}^{hx} \Bigg|_{y=0}^{hy}$$

$$= \frac{\Delta x\, \Delta y}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\int_{Se} \frac{\partial \overline{N}}{\partial y}\, dse = \frac{\Delta x}{2} \begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}$$

FEM (Weighted Residuals)

Write E_z, H_x, H_y as sum of basis fn's

$$E_z(xyt) = \sum_{i=1}^{M_{ez}} N_{ezi}\, E_{zi}(t)$$

H_x

H_y

Rewrite in terms of vector eqn

$$E_z(xyt) = \bar{N}_{ez}\, \bar{E}_{zi}$$

$$= \begin{bmatrix} N_{ez1} & N_{ez2} & \dots & N_{ezm} \end{bmatrix} \begin{bmatrix} E_{z1} \\ \vdots \\ E_{zm} \end{bmatrix}$$

Substitute into 2D Maxwell's eqns

$$-\mu \frac{\partial \bar{N}_{Hx} \bar{H}_{xi}}{\partial t} - \frac{\partial \bar{N}_{Ez} \bar{E}_{zi}}{\partial y} = 0$$

$-\mu$

$$\left(\sigma + \varepsilon \frac{\partial}{\partial t}\right) \bar{N}_{Ez}\, \bar{E}_{zi} - \frac{\partial \bar{N}_{Hy} \bar{H}_{yi}}{\partial x} + \frac{\partial \bar{N}_{Hx} \bar{H}_{yi}}{\partial y} = 0$$

This is one eqn & $N_{ez} + N_{Hy} + M_{Hx}$ unknowns

$$\int_{se} \frac{\partial \bar{N}}{\partial x}\, dse = \frac{\Delta y}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix}$$

$$\sum_{\#elem} \int_{se} \bar{W}\left[\mu \frac{\partial}{\partial t}\bar{N}\,\bar{H}_{xi} + \frac{\partial}{\partial y}\bar{N}\;\bar{E}_{zi}\right] dse = 0$$

$\bar{W} = 1$ (pulse)

$$\sum_{\#elem}\left[\mu \left(\int_{se} \bar{N}\, dse\right)\frac{\partial \bar{H}_{xi}}{\partial t} + \left(\int_{se}\frac{\partial \bar{N}}{\partial y}\, dse\right)\bar{E}_{zi}\right] = 0$$

$$\sum_{\#elem}\left[\mu \frac{\Delta x \Delta y}{4}[1\,1\,1\,1]\,\frac{\partial \bar{H}_{xi}}{\partial t} + \frac{\Delta x}{2}[-1\,-1\,\,1\,\,1]\,\bar{E}_{z1}\right] = 0$$

For one element we get

$$0 = \mu \frac{\partial}{\partial t}\underbrace{\left[\frac{H_{x1} + H_{x2} + H_{x3} + H_{x4}}{4}\right]}_{\substack{\text{Average of } Hx \rightarrow \\ Hx \text{ @ cell centr}}} + \underbrace{\frac{(E_{z4} - E_{z1}) + (E_{z3} - E_{z2})}{2\Delta y}}_{\substack{\text{Av } \partial E_z/\partial y \rightarrow \\ \partial E_z/\partial y \text{ @ cell ctr.}}}$$

(Similarly for other 2 eqⁿs)

Time derivatives in FE-TD are handled exactly like they are in FD-TD.

Using a square cell is exactly like averaging to center of the cell... linear interpolation.