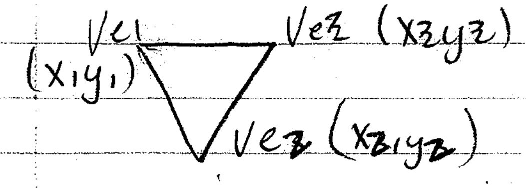
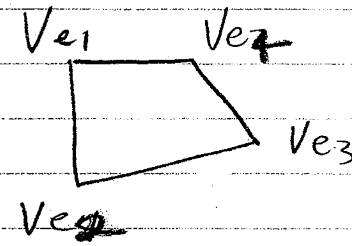
FINITE ELEMENT

$$V(x,y) \cong \sum_{e=1}^{N} V_e(x,y) = \sum_{e=1}^{N} V(x,y)\, u_e(x,y)$$





$$V_e(x,y) = a + bx + cy \qquad V_e(x,y) = a + bx + cy + dxy$$

$= 0$ outside element $\qquad = 0$ outside element

$$\bar{E}_e = -\nabla V_e = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right) V_e$$

$$= -b\hat{x} - c\hat{y} = \text{constant}$$

$\therefore$ Potential V varies linearly within element

Electric field E is constant within element

GOVERNING EQUATIONS ("STIFFNESS MATRIX")

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

unknowns

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

$$Ve(x,y) = a + bx + cy = [1 \; x \; y]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= [1 \; x \; y]\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1}\begin{bmatrix} Ve_1 \\ Ve_2 \\ Ve_3 \end{bmatrix}$$

$$= \frac{1}{2A}\underbrace{\underset{1\times 3}{[1 \; x \; y]}\underset{3\times 3}{\begin{bmatrix} \overset{\swarrow a_1}{(x_2y_3 - x_3y_2)} & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ \overset{\swarrow a_2}{(y_2 - y_3)} & (y_3 - y_1) & (y_1 - y_2) \\ \overset{\swarrow a_3}{(x_3 - x_2)} & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}}}\begin{bmatrix} Ve_1 \\ Ve_2 \\ Ve_3 \end{bmatrix}$$

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$N_1 = \alpha_1^{(e)}(x,y) = \frac{1}{2A}[a_1 + b_1x + c_1y]$

$N_2 = \alpha_2^{(e)}(x,y) = \frac{1}{2A}[a_2 + b_2x + c_2y]$

$N_3 = \alpha_3^{(e)}(x,y) =$

$$Ve(x,y) = \underset{1\times 3}{[\alpha_1 \; \alpha_2 \; \alpha_3]}\begin{bmatrix} Ve_1 \\ Ve_2 \\ Ve_3 \end{bmatrix} = \sum_{i=1}^{3} \alpha_i(x,y)\, Ve_i$$

ELEMENT SHAPE F^ns

α_i's are linearly interpolating f^ns between the nodes.

$$\alpha_i(x_i, y_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \Big\} \text{@ nodes}$$

$$\sum_{i=1}^{3} \alpha_i(x,y) = 1$$

ENERGY

The energy per unit length of the element is

$$W_e = \frac{1}{2}\int \varepsilon |E|^2 \, ds = \frac{1}{2}\int \varepsilon |\nabla V_e|^2 \, ds$$

$$\nabla V_e = \frac{\partial V_e}{\partial x}\hat{x} + \frac{\partial V_e}{\partial y}\hat{y} \qquad \frac{\partial}{\partial z} = 0 \text{ Because 2D}$$

$$= \frac{\partial}{\partial x}\sum_{i=1}^{3} V_{ei}\alpha_i \, \hat{x} + \frac{\partial}{\partial y}\sum_{i=1}^{3} V_{ei}\alpha_i \, \hat{y}$$

$$= \sum_{i=1}^{3} V_{ei} \nabla \alpha_i$$

$$W_e = \frac{1}{2}\int \varepsilon \left(\sum_{i=1}^{3} V_{ei}\nabla\alpha_i\right)^2 ds = \frac{1}{2}\int \varepsilon \sum_{i=1}^{3} V_{ei}\alpha_i \sum_{j=1}^{3} V_{ej}\alpha_j \, ds$$

$$= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon V_{ei} \underbrace{\left[\int \nabla\alpha_i \cdot \nabla\alpha_j \, ds_e\right]}_{C_{ij}^{(e)}} V_{ej}$$

$$C_{ij}^{(e)} = \int \nabla\alpha_i \cdot \nabla\alpha_j \, ds_e$$

$$\nabla\alpha_1 = \frac{1}{2A}\left[(y_2 - y_3)\hat{x} + (x_3 - x_2)\hat{y}\right]$$

$$\nabla\alpha_2 = \frac{1}{2A}\left[(y_3 - y_1)\hat{x} + (x_1 - x_3)\hat{y}\right]$$

$$\nabla\alpha_3 = \frac{1}{2A}\left[(y_1 - y_2)\hat{x} + (x_2 - x_1)\hat{y}\right]$$

$$C_{11}^{(e)} = \nabla\alpha_1 \cdot \nabla\alpha_1 = \frac{1}{4A^{\cancel{2}}}\left[(y_2-y_3)^2 + (x_3-x_2)^2\right]\cancel{\int dS}^{A}$$

$$= \frac{1}{4A}\left[(y_2-y_3)^2 + (x_3-x_2)^2\right]$$

$$\bar{\bar{C}}^{(e)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

↑ COEFFICIENT MATRIX
STIFFNESS MATRIX } C_{ij} Coupling between nodes i, j

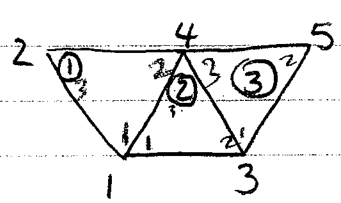
$$\bar{V}_e = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

$$W_e = \frac{1}{2}\varepsilon\, \bar{V}_e^T\, \underset{\uparrow \text{elemental}}{\bar{\bar{C}}^{(e)}}\, \bar{V}_e$$

ASSEMBLE ELEMENTS

TOTAL ENERGY $\quad W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon\, \bar{V}^T\, \underset{\uparrow \text{global}}{\bar{\bar{C}}}\, \bar{V}$

$$\bar{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$



	Global		Local
①	1 - 4 - 2	⇒	1-2-3
②	1 - 3 - 4	⇒	1-2-3
③	3 - 5 - 4	⇒	1-2-3

element 1

Nodes 1 & 1 coupling

$$W = \tfrac{1}{2}\varepsilon\,[V_1\ V_4\ V_2]\ \begin{array}{c} \\ 1 \\ 4 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 4 & 2 \end{array} \\ \begin{bmatrix} C_{11}^{(1)} & C_{14} & C_{12} \\ C_{41} & C_{44} & C_{42} \\ C_{21} & C_{24} & C_{22} \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_4 \\ V_2 \end{bmatrix}$$

$$+\tfrac{1}{2}\varepsilon\,[V_1\ V_3\ V_4]\begin{bmatrix} C_{11} & C_{13} & C_{14} \\ C_{31} & C_{33} & C_{34} \\ C_{41} & C_{43} & C_{44} \end{bmatrix}\begin{bmatrix} V_1 \\ V_3 \\ V_4 \end{bmatrix}$$

$$+\tfrac{1}{2}\varepsilon\,[V_3\ V_5\ V_4]\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} V_3 \\ V_5 \\ V_4 \end{bmatrix}$$

$$= \tfrac{1}{2}\varepsilon\Big[(V_1C_{11}^{(1)} + V_4C_{41} + V_2C_{21})(\qquad)(\qquad)\Big]\begin{bmatrix} V_1 \\ V_4 \\ V_2 \end{bmatrix}$$

$$+\tfrac{1}{2}\varepsilon\Big[(V_1C_{11}^{(2)} + V_3C_{31} + V_4C_{41})(\qquad)(\qquad)\Big]\begin{bmatrix} V_1 \\ V_3 \\ V_4 \end{bmatrix}$$

$+$

$$= \tfrac{1}{2}\varepsilon\,(V_1C_{11}^{(1)}V_1 + \dots)$$
$$+\tfrac{1}{2}\varepsilon\,(V_1C_{11}^{(2)}V_1 + \dots)$$
$$+ \dots$$

$$= \tfrac{1}{2}\varepsilon\,V_1\big(C_{11}^{(1)} + C_{11}^{(2)}\big)V_1 + \dots$$

$C_{11}^{(1)} + C_{11}^{(2)}$

$$= \tfrac{1}{2}\varepsilon\,[V_1\ V_2\ V_3\ V_4\ V_5]\begin{bmatrix} C_{11}' & C_{12}' & C_{13}' & C_{14}' & C_{15}' \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

$$C_{11}' = C_{11}^{(1)} + C_{11}^{(2)}$$

$$C_{11}^{(e)} = \frac{1}{4A_e}\left[\left(\underset{\text{local}}{y_{\overset{\nearrow 4 \text{ Global}}{2}}} - y_{\overset{\nearrow 2}{3}}\right)^2 + \left(x_{\overset{\nearrow 2}{3}} - x_{\overset{\nearrow 4}{2}}\right)^2\right]$$

$$C_{11}^{(1)} = \frac{1}{4A_1}\left[(y_4 - y_2)^2 + (x_2 - x_4)^2\right]$$

$$C_{11}^{(2)} = \frac{1}{4A_2}\left[(y_3 - y_4)^2 + (x_4 - x_3)^2\right]$$

For Global matrix:

$$C_{ij}' = \Sigma_1 C_{ij}^{(e)}$$

all elements containing nodes i & j

$$[C] = \left[\begin{matrix} (C_{11}^{(1)} + C_{11}^{(2)}) & C_{13}^{(1)} & C_{12}^{(2)} & (C_{12}^{(1)} + C_{13}^{(2)}) \\ & & & \\ & & & \end{matrix}\right.$$

C_{11}' NODES 1,1 : $C_{11}^{(1)} + C_{11}^{(2)}$ = Σ elements containing both nodes

C_{12}' 1,2 : $C_{13}^{(1)}$

↑ element 1 node 3 = global node 2

C_{13}' 1,3 : $C_{12}^{(2)}$

1,4 : $C_{12}^{(1)} + C_{13}^{(2)}$

↑ el 1 node 2 ↑ el 2 node 3 = global node 4

1,5 : 0

PROPERTIES OF [C]

① There are 9 coefficients in each element matrix.
There are 3 elements.
∴ There are 27 coefficients...

② Symmetric $C_{ij}^{(e)} = C_{ji}^{(e)}$

③ For a large # of elements, [C] is sparse + banded

④ Singular (ill conditioned)

<u>MINIMIZE ENERGY WRT UNKNOWNS</u>

→ Soln to Laplace's Eqn

$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_N} = 0$$

$$W = \frac{1}{2}\varepsilon \begin{bmatrix} (V_1C_{11} + V_2C_{12} + V_3C_{13} + \cdots) \\ (V_1C_{21} + \\ (V_1C_{31} + \\ (V_1C_{41} + \\ (V_1C_{51} + \end{bmatrix}^T \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

$$= \frac{1}{2}\varepsilon\Big[(V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15})V_1 + (V_1C_{21})V_2 + \cdots + (V_1C_{31})V_3 + \cdots + V_1C_{41}V_4 + V_1C_{51}V_5 + \cdots\Big]$$

equal

$$\frac{\partial W}{\partial V_1} = \frac{1}{2}\varepsilon\Big[2V_1C_{11} + V_2C_{21} + V_3C_{13} + V_4C_{14} + V_5C_{15} + V_2C_{21} + V_3C_{31} + V_4C_{41} + V_5C_{51}\Big]$$

$$\frac{\partial W}{\partial V_1} = \frac{1}{2}\varepsilon\left[\begin{array}{l} 2V_1C_{11} + 2V_2C_{12} + 2V_3C_{13} \\ + 2V_4C_{14} + 2V_5C_{15} \end{array}\right] = 0$$

$$0 = V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15}$$

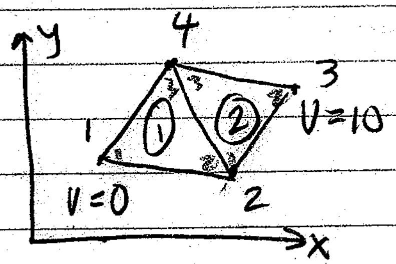
In general: $0 = \sum_{i=1}^{N} V_i C_{ki} = \sum_{i=1}^{N} V_i C_{ik}$

solve for these

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply BC

Example



Node	x	y
1	0.8	1.8
2	1.4	1.4
3	2.1	2.1
4	1.2	2.7

Element 1:

1 1-2-4 = (1-2-3)

$P_1 = y_2 - y_3^{\,4} = -1.3$ $\quad Q_1 = x_3^{\,4} - x_2 = -.2$

$P_2 = y_3^{\,4} - y_1 = 0.9$ $\quad Q_2 = x_1 - x_3^{\,4} = -.4$

$P_3 = y_1 - y_2 = 0.4$ $\quad Q_3 = x_2 - x_1 = 0.6$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 0.35$$

$$C_{ij}^{(e)} = \frac{1}{4A}\left[P_iP_j - Q_iQ_j\right]$$

$$\bar{\bar{C}}^{(1)} = \begin{bmatrix} 1.236 & -0.7786 & -.4571 \\ -.7786 & .6929 & .0857 \\ -.4571 & .0857 & .3714 \end{bmatrix}$$

Element 2

② 2-3-4 (1-2-3)

$P_1 = -.6$ $Q_1 = -.9$

$P_2 = 1.3$ $Q_2 = .2$

$P_3 = -.7$ $Q_3 = .7$

$A = .525$

$$\bar{\bar{C}}^{(2)} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Assemble global matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$C_{11} = C_{11}^{(1)}$	$C_{21} = C_{12}$	$C_{31} = C_{13}$
$C_{12} = C_{12}^{(1)}$	$C_{22} = C_{22}^{(1)} + C_{11}^{(2)}$	$C_{32} = C_{23}$
$C_{13} = 0$	$C_{23} = C_{12}^{(2)}$	$C_{33} = C_{22}^{(2)}$
$C_{14} = C_{14}^{(1)}$	$C_{24} = C_{23}^{(1)} + C_{13}^{(2)}$	$C_{34} = C_{23}^{(2)}$

$C_{41} = C_{14}$ $C_{43} = C_{34}$

$C_{42} = C_{24}$ $C_{44} = C_{33}^{(1)} + C_{33}^{(2)}$

But V_1 & V_3 are fixed.

Band Matrix Method:

(Divide free + fixed nodes
→ nonsingular matrix)

$$W = \frac{1}{2}\varepsilon \begin{bmatrix} V_f & V_p \end{bmatrix} \begin{bmatrix} C_{FF} & C_{FP} \\ C_{PF} & C_{PP} \end{bmatrix} \begin{bmatrix} V_f \\ V_p \end{bmatrix}$$

prescribed (fixed)

free

$$\nabla V_p = 0$$

$$\frac{\partial W}{\partial V_i} = 0 \ldots$$

$$\begin{bmatrix} C_{FF} & C_{FP} \end{bmatrix} \begin{bmatrix} V_F \\ V_P \end{bmatrix} = 0$$

$$\bar{\bar{C}}_{FF} \bar{V}_F = - \bar{\bar{C}}_{FP} \bar{V}_P$$

$$\bar{\bar{A}} \ \bar{x} = \bar{b}$$

C_{FF} $\quad$ C_{FP} $\quad$ V_F

Row 2

$$\left[\begin{array}{cc|cc} C_{22} & C_{24} & C_{21} & C_{23} \\ C_{42} & C_{44} & C_{41} & C_{43} \\ \hline C_{12} & C_{14} & C_{11} & C_{13} \\ C_{32} & C_{34} & C_{31} & C_{33} \end{array}\right] \left[\begin{array}{c} V_2 \\ V_4 \\ \hline V_1 \\ V_3 \end{array}\right] = 0 \qquad \left.\begin{array}{c} V_1 \\ V_3 \end{array}\right\} \text{Known}$$

C_{PF} $\quad$ C_{PP} $\quad$ V_P

$$C_{22}V_2 + C_{24}V_4 + C_{21}V_1 + C_{23}V_3$$

$$\begin{bmatrix} C_{22} & C_{24} \\ C_{42} & C_{44} \end{bmatrix} \begin{bmatrix} V_2 \\ V_4 \end{bmatrix} = -\begin{bmatrix} C_{21} & C_{23} \\ C_{41} & C_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix}$$

Not singular

$$= \begin{bmatrix} -C_{21}V_1 - C_{23}V_3 \\ -C_{41}V_1 - C_{43}V_3 \end{bmatrix}$$

$$\bar{\bar{C}}_{FF} \; \bar{V}_F = -\bar{\bar{C}}_{FP} \; \bar{V}_P$$

$$\bar{\bar{A}} \; \bar{x} = \bar{b}$$

In general:

You must prescribe 2 boundary conditions

a) V_i = fixed

b) ∇V_i = fixed

FEM for Laplace's Eq^{n}

$I(\phi) =$ Equivalent Functional

$$= \int_S \nabla^2 \phi \, dS = \int_S |\nabla \phi|^2 \, dS$$

$$= \underbrace{\sum_e}_{\#\text{ of elements}} I_e(\phi)$$

Assume:

$$\underbrace{\phi_e(x,y)}_{\text{potential in each element}} = \sum_{i=1}^{\overset{\#\text{ of nodes in element}}{3}} \underbrace{\alpha_i(x,y)}_{\text{elemental shape } f^{n}} \underbrace{\phi_i^e}_{\text{potential @ nodes}}$$

$$\nabla_{2D} \phi = \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y}$$

$$= \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} \alpha_i(x,y)\,\phi_i^e\right)\hat{x} + \frac{\partial}{\partial y}\left(\quad \right)\hat{y}$$

$$= \sum_{i=1}^{3}\left(\frac{\partial \alpha_i(x,y)}{\partial x}\hat{x} + \frac{\partial \alpha_i(x,y)}{\partial y}\hat{y}\right)\phi_i^e$$

$$= \begin{bmatrix} \frac{\partial \alpha_1}{\partial x} & \frac{\partial \alpha_2}{\partial x} & \frac{\partial \alpha_3}{\partial x} \\ \frac{\partial \alpha_1}{\partial y} & \frac{\partial \alpha_2}{\partial y} & \frac{\partial \alpha_3}{\partial y} \end{bmatrix} \begin{bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{bmatrix}$$

$\alpha_i = (a_i + b_i x + c_i y)\frac{1}{2A}$ see eq^{n} (8) in handout

$$\nabla\phi = \underbrace{\frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}}_{\overline{\overline{B}}^e} \underbrace{\begin{bmatrix} \phi_{e1} \\ \phi_{e2} \\ \phi_{e3} \end{bmatrix}}_{\overline{\phi}e} = \overline{\overline{B}}_e(\phi)$$

$$I_e(\phi) = \int_{S_e} \{\nabla\phi\}^T \{\nabla(\phi)\}\, dx\,dy \quad \text{where } \nabla\phi = \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{bmatrix}$$

$$= \int_{S_e} \overline{\overline{B}}e(\phi)^T\, \overline{\overline{B}}e(\phi)\, dx\,dy$$

$$= \int_{S_e} \underbrace{\left(\overline{\overline{B}}e\, \overline{\Phi}e\right)^T}_{\overline{\Phi}e^T\, \overline{\overline{B}}e^T} \left(\overline{\overline{B}}e\, \overline{\Phi}e\right) dx\,dy$$

$$= \overline{\phi}e^T \underbrace{\left[\int_{S_e} \overline{\overline{B}}e^T\, \overline{\overline{B}}e\, dx\,dy\right]}_{\overline{\overline{C}}^e} \overline{\Phi}e$$

$$\overline{\overline{C}}^e = \frac{1}{4A}\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$I_e(\phi) = \overline{\phi}e^T\, \overline{\overline{C}}e\, \overline{\Phi}e$$

Combine elemental coefficient matrices into global coefficient matrix

$$I_e(\phi) = \overline{\phi}^T\, \overline{\overline{C}}\, \overline{\phi}$$

Minimize functional

$$\frac{\partial I(\phi)}{\partial \phi_i} = 0 \rightarrow \bar{\bar{C}}\,\bar{\Phi} = 0$$

FEM for 2D Poisson Eqn

$$I(\phi) = \int_S \left(\nabla^2\phi + \frac{\rho\phi}{\varepsilon}\right) dx\,dy$$

$$= \int_s |\nabla\phi|^2 dx\,dy + \int_S \frac{\rho}{\varepsilon}\phi\, dx\,dy$$

$$= \underbrace{I_1(\phi)}_{\text{from Laplace's eqn}} + I_2(\phi) = \sum_{e=1}^{Ne}\left[I_{e1}(\phi) + I_{e2}(\phi)\right]$$

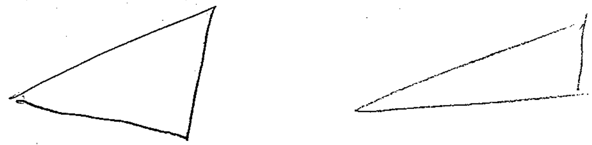
$$I_{e2}(\phi) = \int_{se} \frac{\rho}{\varepsilon}\left(\alpha_1\Phi_1 + \alpha_2\Phi_2 + \alpha_3\Phi_3\right) dx\,dy$$

$$\frac{\partial I_2(\Phi)}{\partial \phi} = \sum_{e=1}^{Ne} \frac{\partial I_{e2}(\Phi)}{\partial \phi} = \sum_{e=1}^{Ne} \begin{bmatrix} \partial/\partial\Phi_1 \\ \partial/\partial\Phi_2 \\ \partial/\partial\Phi_3 \end{bmatrix} I_{e2}(\Phi)$$

assume constant within element

$$= \sum_{e=1}^{Ne} \int_{Se} \frac{\rho}{\varepsilon} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} dx\,dy$$

$$= \sum_{e=1}^{Ne} \frac{\rho_e}{\varepsilon_e} \int_{se} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} dx\,dy \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{A}{3}$$



$$\int_{S_e} \alpha_1 \, dx\,dy = \int_{S_e} (a_1 + b_1 x + c_1 y) \frac{dx\,dy}{2A}$$

Remember: @ $\alpha_1(x_1 y_1) = 1$

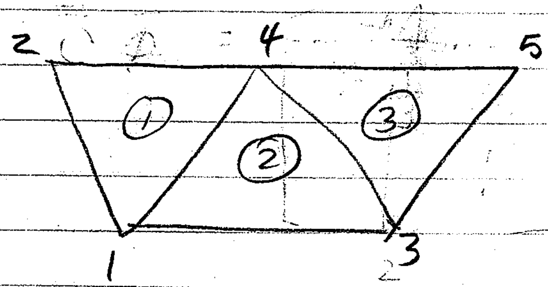
$\alpha_1(x_2 y_2) = \alpha_1(x_3 y_3) = 0$

Area under curve $= \dfrac{A * 1}{3}$

1/3 of space raised

$$\frac{\partial I_{2e}(\phi)}{\partial \phi} = \sum_{e=1}^{N_e} \frac{\rho_e}{\varepsilon_e} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{A_e}{3}$$

$$\frac{\partial I}{\partial \phi} = \frac{\partial I_1(\phi)}{\partial \phi} + \frac{\partial I_2(\phi)}{\partial \phi} = 0 \quad \text{minimize}$$



$$P_e = \frac{\rho_e A_e}{3\varepsilon_e}$$

$$\frac{\partial I_2(\phi)}{\partial \phi} = \begin{bmatrix} P_1 + P_2 \\ P_1 \\ P_2 + P_3 \\ P_1 + P_2 + P_3 \\ P_3 \end{bmatrix}$$