ECE 5340/6340 SOR: Successive Over-Relaxation Method

ITERATIVE METHODS OF SOLVING MATRIX EQUATIONS:

Particularly good for solving sparse matrix equations (Finite Element method and Finite Difference Method)

Solve A x = b

Back Substitution Algorithm:

$$x_i = \frac{\sum_{j=1}^{n} a_{ij} x_j + b_i}{a_{ii}}$$ For i=1,2,3,…n

In regular back substitution, we know x_j. But what if we didn't? We could guess! These iterative methods are based on how to choose and improve that guess.

Jacobi's method

Initial guess: $x^{(0)} = 0$

Then at each (k^{th}) iteration find the next $(k+1)^{th}$ values of x:

$$x_i^{(k+1)} = \frac{-\sum_{j=1}^{n} a_{ij} x_j^{(k)} + b_i}{a_{ii}} \quad for\_i = 1,2,\ldots,n$$

For a banded matrix, this summation can be limited to the bands

For the Jacobi method, new (k+1) values are not used until the next iteration.

Gauss-Seidel

This method improves on the Jacobi method by using new values that have been obtained prior to each step in the iteration. This gives faster convergence.

$$x_i^{(k+1)} = \frac{-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + b_i}{a_{ii}}$$

New values are used as soon as they are generated.

SOR: Successive Over-Relaxation

Relaxation moves towards solution faster:

$$x_i^{(k+1)} = x_i^{(k)} + \omega R_i$$

From Gauss-Seidel:

$$x_i^{(k+1)} = \frac{-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + b_i - \sum_{j=i}^{i} a_{ij} x_j^{(k)} + \sum_{j=i}^{i} a_{ij} x_j^{(k)}}{a_{ii}}$$

$$= x_i^{(k)} - \frac{\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} + \sum_{j=i}^{n} a_{ij} x_j^{(k)} + b_i}{a_{ii}}$$

This can be written: $x_i^{k+1} = x_i^{k} + R_i$

Where R_i is the "Residual" (error or change)

Now, use relaxation (ω) to speed convergence:

$$x_i^{(k+1)} = x_i^{(k)} + \omega R_i$$

How to choose ω :

1. For $\omega = 1$, this reduces to Gauss-Seidel
2. Method converges when $0 < \omega < 2$ for a positive-definite matrix. (When matrix is reduced to diagonal, all elements are positive.)
3. $0 < \omega < 1$ Under-relaxation slows convergence
4. $1 < \omega < 2$ Over-relaxation speeds convergence
5. ω optimal is based on spectral radius, which is difficult (expensive) to calculate.
6. For square matrices, ω optimal can be approximated:

$$\omega = 4 / (2 + \sqrt{(4 + C*C)})$$

$$C = \cos(\pi / p) + \cos(\pi / q)$$

p,q = # of mesh divisions on x,y sides

EXAMPLE:

See web