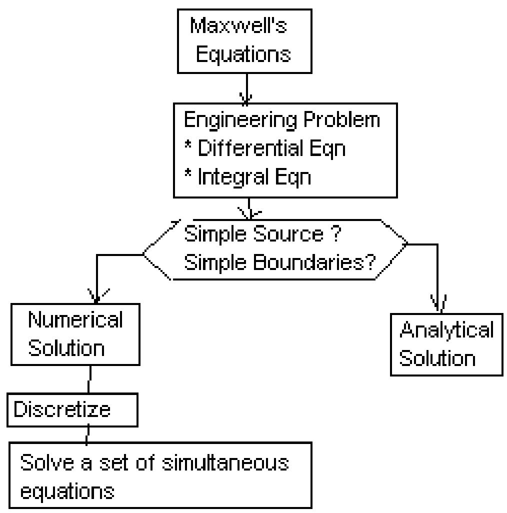
ECE 5340/6340: Lecture 4 -- REVIEW OF MATRIX ALGEBRA

Why matrix equations are important in numerical methods:



SIMULTANEOUS EQUATIONS: are of the form

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + a_{2n} x_n = b_2$$

$$a_{m1}x_1 + a_{m2} x_2 + a_{m3} x_3 + \dots + a_{mn} x_n = b_m$$

Which can be written as a matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix}$$

size:

mxn $\qquad$ $nx1$ $\quad$ $mx1$

$$\bar{\bar{A}}\bar{x} = \bar{b}$$

MATRIX ADDITION:

$$\overline{\overline{A}} + \overline{\overline{B}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{(2x2)} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{(2x2)} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}_{(2x2)}$$

MATRIX TRANSPOSE:

$$\vec{\overline{A}} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}_{(3x1)} \qquad \overline{\overline{A}}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \end{bmatrix}_{(1x3)}$$

$$\vec{\overline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{(3x2)} \qquad \overline{\overline{A}}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{(2x3)}$$

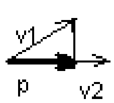
INNER OR DOT PRODUCT

$$\overline{\overline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{(2x3)} \qquad \overline{\overline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{(3x2)}$$

$$\overline{\overline{A}} \bullet \overline{\overline{B}} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) \end{bmatrix}_{(2x2)}$$

VECTOR DOT PRODUCT

$\mathbf{p} = \mathbf{v}_1 \bullet \mathbf{v}_2$ = component of $\mathbf{v}_1$ in $\mathbf{v}_2$ direction = $|\mathbf{v}_1||\mathbf{v}_2|\cos(\alpha)$



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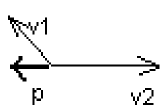

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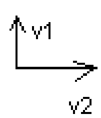


p=0

(α = angle between vectors)

DETERMINANT of a matrix (as it relates to matrix singularity)

$2D$:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$3D$:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

4D: Repeat.

Method:
1) Find minor matrices by striking row and column.
2) Multiply by element
3) Signs alternate +-+-+- …

Determinant: Defines the "hypervolume" of a matrix.

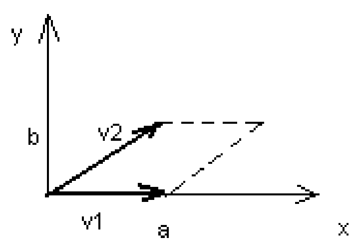
2x2 : area defined by parallelogram of matrix vectors

3x3 : volume defined by parallelopiped of matrix vectors

$$\bar{v}_1 = a\hat{x}$$

$$\bar{v}_2 = a\hat{x} + b\hat{y}$$

$$Area = \begin{vmatrix} \bar{v}_1 \\ \bar{v}_2 \end{vmatrix} = \begin{vmatrix} a & 0 \\ a & b \end{vmatrix} = ab$$



- If any two vectors become coincident (2D: parallel / 3D: in the same plane), then the area or volume collapses to zero.
- If you are solving 3 equations in 3 unknowns, you can only solve it if your equations (vectors) are independent (not coincident).
- Thus, if the determinant of the matrix is zero (or near zero), you cannot solve the matrix equation.
- This is called a **"singular matrix"**.

DIAGONAL MATRIX

$$\overline{\overline{D}} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$\left|\overline{\overline{D}}\right| = d_1 d_2 d_3$$

IDENTITY MATRIX

$$\overline{\overline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left|\overline{\overline{I}}\right| = 1 \quad (\textit{for any size})$$

TRIANGULAR MATRIX

$$LowerTriangular = \begin{bmatrix} d_1 & 0 & 0 \\ t_1 & d_2 & 0 \\ t_2 & t_3 & d_3 \end{bmatrix}$$

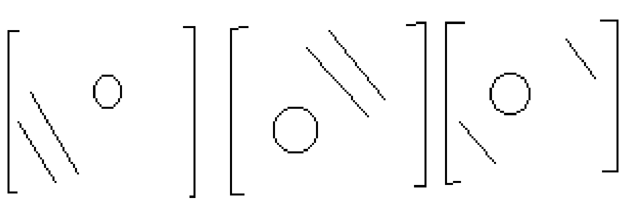
$$UpperTriangular = \begin{bmatrix} d_1 & t_1 & t_2 \\ 0 & d_2 & t_3 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$|UT| = |LT| = d_1 d_2 d_3$$

DETERMINANT OF PRODUCT OF MATRICES:

$$\left|\overline{\overline{A}}\,\overline{\overline{B}}\,\overline{\overline{C}}\right| = \left|\overline{\overline{A}}\right|\left|\overline{\overline{B}}\right|\left|\overline{\overline{C}}\right|$$

BANDED MATRIX:



SPARSE MATRIX:

$$\begin{bmatrix} x & & x & \\ & & & \\ & & x & \\ & x & & \end{bmatrix}$$

VECTOR CROSS PRODUCT

$$\overline{A} \times \overline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\overline{A}||\overline{B}|\sin\alpha$$