Moment Methods in Electromagnetics for Undergraduates

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Abstract—Moment methods have recently emerged as one of the most important tools for the solution of electromagnetic fields problems. This approach is conceptually simple and easily usable by the practicing engineer. However, the formal presentation of moment method is founded in the theory of linear vector spaces which makes it beyond the scope of most undergraduate electrical engineering curricula. This paper documents the experience in undergraduate teaching of moment ·thods in electromagnetics through the solution of integral equations a "point matching" or collocation approach at the University of ssissippi. The main purpose of the paper is to provide a guide for teaching moment methods for the educator who is involved with undergraduate fields courses, particularly, the non-field theorist. Static and time-harmonic examples are presented with enough detail for in-course use.

I. Introduction

IN recent years a considerable amount of interest has been de d by the electromagnetic community to the application of the moment method [1] for solution of boundary value problems. The principal reason for the attraction to this numerical matrix approach is the tremendous versatility it offers in the treatment of structures of arbitrary configurations. Thus, problems which were hitherto untractable by classical approaches, such as the separation of variables method, are now routinely handled, as evidenced by the steady stream of papers using moment methods which now appear regularly in the literature. Besides its flexibility, the moment method has the advantage that it is conceptually simple and from an applications viewpoint is devoid of complicated mathematics: thus, this approach is readily usable by a large group of the electrical engineering community. It is therefore not difficult to envision the moment method becoming even more popular and, in the future, constituting one of the most important tools for analysis of electromagnetic problems.

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At its inception, the moment method was developed primarily for the researcher [1]. As a consequence, explanations of its foundations were based on concepts from the theory of linear vector spaces, with the result that it is beyond the scope of most undergraduate curricula. Because of its numerous advantages which have led to its rapid adoption by applications engineers in the field, the need to incorporate the moment method in undergraduate courses is gradually becoming evident. In fact, one recent undergraduate electromagnetics textbook [2] devotes two chapters to the treatment of electrostatics problems by matrix methods. While the advanced researcher finds the formal development of moment method through linear vector space theory straightforward, most undergraduate and beginning graduate students at present educational levels find the transition awkward. Specifically, no pedagogical algorithm exists to help the undergraduate student to relate the procedures of the moment method to either his intuition or former training.

In this paper, a technique for presenting the moment method in elementary terms is developed. The vehicle employed is the electrostatics problem of determining the charge distribution on a thin wire held at a constant potential. The only prerequisites necessary are elementary physics and calculus concepts. The development will be seen to evolve from familiar basic circuit ideas and integral calculus interpretations. The experiences of the Electrical Engineering Faculty at the University of Mississippi with this teaching experiment will be reported. Other sample problems are also supplied.

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II. The Static Charge Distribution on a Thin Wire at a Constant Potential

The ideas of the moment method will be introduced at an elementary level in this section through the example problem of determining the charge distribution on a constant potential wire. Usually, a beginning undergraduate electromagnetic field course starts by establishing electrostatics concepts. The notions of charge distributions, $\rho(\bar{r}')$, giving rise to potentials, $\Phi(\bar{r})$, from which fields may be determined should already be familiar. Thus, the potential due to an electrostatic line source is

$$\Phi(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_{\substack{\text{line}\\ \text{source}}} \frac{\rho(\bar{r}')}{R}\, dl' \qquad (1)$$

where ϵ_0 is the permitivity of free space, l' is the distance measured along the line source, $\bar{r} = (x, y, z)$ denotes the observation coordinates, and $\bar{r}' = (x', y', z')$ denotes the source coordinates with the distance from (x, y, z) to (x', y', z') given by

$$R = |\bar{r} - \bar{r}'| = ((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2} \qquad (2)$$

using conventional geometric descriptions. Typical uses of this relationship are, for example, determining potentials and then fields due to an infinitely long line charge or to a circular loop on which the charge distribution is constant. The inquisitive student, however, may question the usefulness of these idealized problems. Specifically, how in practice does one establish a constant charge distribution? If a battery is connected to a wire does the resulting charge assume a constant distribution? The question may thus be posed of how does one actually determine what the charge distribution is in a practical problem. Hence, the stage is set for the introduction of moment methods, through which a myriad of problems may be solved.

Consider a finite length, straight, conducting thin wire of radius, a, situated in free space to which a constant potential of one volt is applied [3] as illustrated in Fig. 1. Because the wire is conducting, charges are free to move, eventually redistributing themselves in some final manner. If we know the charge distribution, then Eq. 1 may be used to compute the potential everywhere. However, it is precisely this charge distribution which is the unknown to be solved for in this problem. Let us therefore seek an alternative interpretation to Eq. 1 in which the right-hand side for this problem is unknown and the left-hand side, Φ, is known. Since the potential everywhere is governed by Eq. 1, let the observation point now fall on the wire where Eq. 1 remains valid. Here the applied potential, which is known, constrains Φ ($\bar{r}$ on wire) to be exactly one volt, and Eq. 1 reduces to an integral equation of the form,

$$1 = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\rho(y')}{|y - y'|}\, dy', \qquad (3)$$

for $-L/2 < y < L/2$ where $\bar{r} \to y\hat{y}$, $\bar{r}' \to y'\hat{y}$, $dl' \to dy'$, and $R \to |y - y'|$. To reiterate, whatever the form of the unknown charge distribution $\rho(y')$, it must satisfy Eq. 3, or, equivalently, it must cause the potential observed everywhere on the wire to be exactly one volt. Eq. 3 thus constitutes an integral equation which can be solved to determine $\rho(y')$ on the wire.

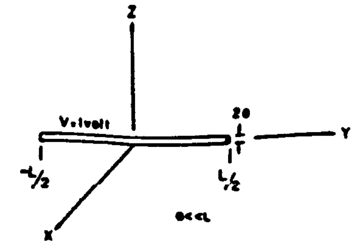

Fig. 1. Finite length wire held at a constant potential.

Let us next seek a numerical solution to this problem. Since Eq. 3 applies for observation points everywhere along the wire, it can be specialized to a fixed point y_k on the wire with the result

$$1 = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\rho(y')}{|y_k - y'|}\, dy'. \qquad (4)$$

Because y_k is constant, the integrand is a function of y' only. The problem is now the determination of this functional dependence. Before proceeding further let us recall a familiar concept from integral calculus. The integral of a function, $f(y)$, may be regarded as the sum of the areas under rectangular strips, each having a height equal to the mean of $f(y)$ over the strip (the essence of numerical integration). Specifically,

$$\int_{-L/2}^{L/2} f(y')\, dy' \cong f(y_1)\Delta y' + f(y_2)\Delta y' + f(y_3)\Delta y' + \cdots + f(y_n)\Delta y' + \cdots + f(y_N)\Delta y'. \qquad (5)$$

Equation 5 applies, of course, when $f(y')$ is a known function but of equal importance it applies even when $f(y')$ is an unknown, if we interpret integrals as merely the area under a curve. Hence, from this interpretation of the potential integral, Eq. 4 may be recast based on Eq. 5 into the form

$$4\pi\epsilon_0 \cong \frac{\rho_1\Delta}{|y_k - y_1'|} + \frac{\rho_2\Delta}{|y_k - y_2'|} + \frac{\rho_3\Delta}{|y_k - y_3'|} + \cdots + \frac{\rho_n\Delta}{|y_k - y_n'|} + \frac{\rho_N\Delta}{|y_k - y_N'|}. \qquad (6)$$

The wire has been divided up into N segments all of length Δ as illustrated in Fig. 2. Over each segment, we can now regard the linear charge density as an unknown constant, ρ_n. The idea is that once these unknown constants, ρ_n's, are determined then the charge distribution over the wire will be specified.

Up to this point, only one equation in terms of N unknown constants has been obtained for the one arbitrary observation point, y_k, (or match point). It is evident that if a solution for these N constants is to follow, then N linearly independent

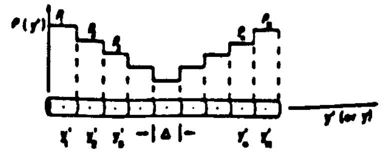

Fig. 2. Division of the wire into segments with individually constant charge distributions (observation or match points can be represented by locations on the y-axis).

equations are required. Because Eq. 6 must hold at all points on the wire we obtain N equations in terms of N unknowns by simply choosing, for convenience, N observation points on the wire as depicted in Fig. 2. The match points y_k are now simply placed in the center of the original Δ's into which the source has been divided (primed coordinates). Applying then Eq. 6 at these N match or observation point locations successively, one obtains a set of N equations as

$$4\pi\epsilon_0 = \frac{\rho_1\Delta}{|y_1 - y_1'|} + \frac{\rho_2\Delta}{|y_1 - y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_1 - y_N'|}$$

$$4\pi\epsilon_0 = \frac{\rho_1\Delta}{|y_2 - y_1'|} + \frac{\rho_2\Delta}{|y_2 - y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_2 - y_N'|}$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$4\pi\epsilon_0 = \frac{\rho_1\Delta}{|y_k - y_1'|} + \frac{\rho_2\Delta}{|y_k - y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_k - y_N'|}$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$4\pi\epsilon_0 = \frac{\rho_1\Delta}{|y_N - y_1'|} + \frac{\rho_2\Delta}{|y_N - y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_N - y_N'|}. \tag{7}$$

Eq. 7 then constitutes the N linear equations which are to be solved for the N unknown constants ρ_n.

The analogy of this system of equations with circuit concepts is obvious and we may write it more succinctly in matrix notation as

$$[l_{mn}][\rho_n] = [V_n] \tag{8}$$

where the l_{mn} terms are $\Delta/|y_m - y_n'|$, the V_n's are $4\pi\epsilon_0$, and ρ_n's, the unknown charge densities. It can now be concluded that, once the matrix equation is solved by any of the several standard inversion or equation solution schemes on a digital computer, the desired charge distribution $\rho(y')$ will be known in discrete form, ρ_n's.

To recapitulate, the solution of the integral equation in Eq. 3 for the charge distribution on a wire at a constant potential has been accomplished first by dividing the wire into constant charge segments and then by successively enforcing Eq. 3 at the centers of these segments. However, the fact that we chose for convenience match points at the centers of the source segments does present a problem. The astute student will observe that, when the match point coinci[illegible] with the source summation index in any one equation [illegible]. 7, i.e., match point equals source point ($y_k = y_n$) or denominator, $|y_k - y_n'| \to 0$, it renders a singular matrix element. For N finite and large, the form of Eq. 6

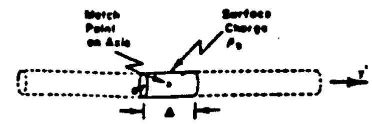

Fig. 3. Geometry for computation of self term.

yields the potential from a collection of N weighted point charges where $Q_n = \Delta\rho_n$, and it is therefore not surprising that we encounter a singularity when the diagonal term or self term is sought because we have really *approximated* the continuous wire as a collection of point charges, Q_n's, and are seeking the potential at the location of one of these charges. ?

Evidently, a more elaborate treatment is needed for the diagonal terms or the potential contribution due to a segment of charge *itself* (the previous treatment has been found to be sufficiently accurate for mutual or non-diagonal terms in most problems) [3].

The wire geometry originally depicted for the electrostatic problem in Fig. 1 shows a finite radius a. The fact that the wire is highly conducting means that a uniform potential exists throughout the wire, which in reality results in a *surface charge density*, ρ_s, over the wire surface. This observation can now be used to compute the self or diagonal terms of the coefficient matrix through a more accurate approximation. With the aid of Fig. 3, the self term may be interpreted as the potential at the center of a uniform tube of an approximate surface charge density ρ_s. Hence,

$$\Phi\,(\text{Tube center}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_{-\Delta/2}^{\Delta/2}\frac{\rho_s a\,d\phi\,dy'}{\sqrt{a^2 + y'^2}}$$

$$= \frac{2\rho_l}{4\pi\epsilon_0}\ln(\Delta/a) \tag{9}$$

where $\rho_l = 2\pi a\rho_s$. Thus, the desired diagonal term (when $m = n$) is

$$l_{nn} = 2\ln(\Delta/a). \tag{10}$$

If one recalls that for $m \neq n$, $l_{mn} = \Delta/|y_m - y_n'|$, he sees that the final matrix equation[1] representing this problem becomes

[1] Each coefficient can be interpreted as the normalized potential at a match point due to a charge source on the n^{th} segment. Hence, the basic sub-problem in this type numerical approach can be defined in terms of the distance measured in reference to a coordinate system localized on the n^{th} source which greatly facilitates the computation of matrix coefficients.

$$\begin{bmatrix} 2\ln(\Delta/a) & \frac{\Delta}{|y_1 - y_2'|} & \frac{\Delta}{|y_1 - y_3'|} & \cdots & \frac{\Delta}{|y_1 - y_N'|} \\ \frac{\Delta}{|y_2 - y_1'|} & 2\ln(\Delta/a) & \frac{\Delta}{|y_2 - y_3'|} & \cdots & \frac{\Delta}{|y_2 - y_N'|} \\ \vdots & \vdots & \vdots & \frac{\Delta}{|y_m - y_n|} & \vdots \\ \frac{\Delta}{|y_N - y_1'|} & \frac{\Delta}{|y_N - y_2'|} & \frac{\Delta}{|y_N - y_3'|} & \cdots & 2\ln(\Delta/a) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \cdot \\ \rho_n \\ \cdot \\ \rho_N \end{bmatrix} = \begin{bmatrix} 4\pi\epsilon_0 \\ 4\pi\epsilon_0 \\ 4\pi\epsilon_0 \\ \cdot \\ 4\pi\epsilon_0 \\ \cdot \\ 4\pi\epsilon_0 \end{bmatrix}. \tag{11}$$

The matrix equation for this problem has been solved using a matrix inversion program, and the results for a sample case where the wire length is 1 meter and the radius is 1 millimeter are presented in Fig. 4 for twenty wire segments (i.e., a 20 X 20 matrix). As can be seen, the charge distribution on the constant potential wire is hardly constant, and it ? exhibits the characteristic singularity at the ends of the wire. This finding together now with some qualitative explanations from the repelling charge viewpoint can hopefully reward the student for his diligence in undertaking the study of this problem. With some guidance, the program logic needed is well within the ability of the typical undergraduate having a basic knowledge of computer programming.

III. Results

Our formal experience with undergraduates solving the static wire moment method problem has been over a period of 5 years in a beginning undergraduate fields course and a problems oriented laboratory course. The student's preparation and background are roughly that found in the statics portion of Hayt's electromagnetic fields text [4]. For the most part, the students have been second semester juniors in a 4-year academic program in electrical engineering. On the average, the response has been quite good with at least 80% of the class being successful in obtaining the correct solution. This was true regardless of how the problem was assigned, i.e., required, optional, and for extra credit (with all methods having been tried). Perhaps this is another instance of the "Hawthorne Uncertainty Principle" as applied to engineering education [5], where if the students know they are being given some new instruction material on an experimental basis, that fact in itself provides motivation for them to cooperate. Several students have elected to continue on to more sophisticated problems in a later senior design course, involving significantly more complex geometries. Some of these type problems are covered in the following examples.

Wire-Type Examples

There are numerous electrostatic problems of the wire type which can be solved with relative ease once the basic approach has been mastered. One such extension of the straight-wire problem of the previous section is the bent-wire problem shown in Fig. 5. The mathematical formulation of this problem is the same as that of the straight wire problem as

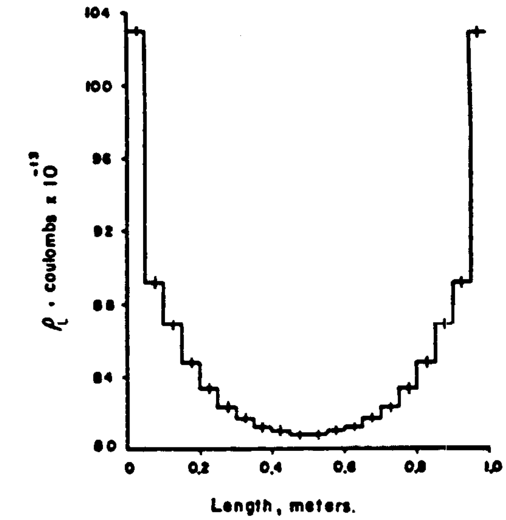

Fig. 4. Line charge density, ρ_L, on a wire of radius 1 mm and length 1 meter at a 1 volt potential (20 unknowns).

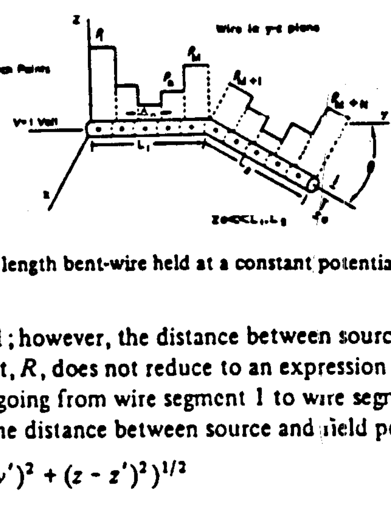

Fig. 5. Finite length bent-wire held at a constant potential of 1 volt.

stated in Eq. 1; however, the distance between source point and field point, R, does not reduce to an expression as simple as $|y - y'|$ in going from wire segment 1 to wire segment 2. In this case, the distance between source and field point is

$$R = ((y - y')^2 + (z - z')^2)^{1/2} \tag{12}$$

and, thus, the potential integral for the bent-wire problem becomes

$$\phi(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_0^{L_1} \frac{\rho(l_1)}{R}\, dl_1 + \int_0^{L_2} \frac{\rho(l_2)}{R}\, dl_2\right] \tag{13}$$

where these are line integrals and l_1 and l_2 are measured along the corresponding wire segment from the left-most end. If one regards these integrals as subareas of rectangular strips as sup- ted in Fig. 5, then for the m^{th} match point on either gment Eq. 13 becomes

$$1 = \frac{1}{4\pi\epsilon_0}\left[\sum_{n=1}^{M} \rho_n \int_{\Delta_n} \frac{dl_1}{R_{mn}} + \sum_{n=M+1}^{N+M} \rho_n \int_{\Delta_n} \frac{dl_2}{R_{mn}}\right] \tag{14}$$

where there are M equal-width rectangular strips on segment 1 and N equal width strips on segment 2. Here, n represents the n^{th} source on either segment 1 or 2 with the related strip width. Δ_n.

The next step in the reduction of Eq. 14 to a system of linear equations is the evaluation of each of the integrals present in this equation for each match point. With this more complex geometry, it has been observed that one cannot use here the simple approximation employed for the integral in the straight wire problem, and, to maintain numerical accuracy, the integrals must be evaluated carefully. The basic sub-problem that is encountered in the integration is presented in Fig. 6. This integration is related to the potential at the m^{th} match point due to a finite-length source of length Δ, and its general form[2] is

$$I_{mn} = \int_{\Delta} \frac{dl}{R_{mn}} = \int_{-\Delta/2}^{\Delta/2} \frac{dl'}{((l_m - l')^2 + z_m^2)^{1/2}} \tag{15}$$

wh reduces to

$$I_{mn} = \begin{cases} 2 \ln (\Delta/a);\ m = n \text{ (i.e., self terms)} \\ \ln \left[\dfrac{d_{mn}^{+} + ((d_{mn}^{+})^2 + z_m^2)^{1/2}}{d_{mn}^{-} + ((d_{mn}^{-})^2 + z_m^2)^{1/2}}\right]; \\ \qquad m \text{ and } n \text{ on different segments} \\ \ln [d_{mn}^{+}/d_{mn}^{-}];\ m \text{ and } n \text{ on same segment} \end{cases} \tag{16}$$

for

R_{mn} = distance between m^{th} match point and a point on n^{th} source,

$d_{mn}^{+} = l_m + \Delta/2$, and $d_{mn}^{-} = l_m - \Delta/2$

where l_m is related to the true distance between the m^{th} field point and the center of the n^{th} source point.

From Eq. 16, we have $[l_{mn}]$ of Eq. 8 for the $M + N$ by $M + N$ matrix equation representing this bent-wire problem having $M + N$ unknown line charge densities. The matrix equation can be solved by standard matrix solution techniques to obtain the ρ_n's for the bent-wire case.

With a 1 meter wire of radius of 1 millimeter bent in the middle such that the angle θ is 90 degrees, then from the solution of the above problem, we find that the charge will distribute itself along the wires as shown in Fig. 7 for $V = 1$ vol One observes that the charge density decreases over the ngth of the bent wire as compared to the straight-wire s would be expected. This solution treats the actual junction problem at the bend of two wires only in a global sense; however, for large length-to-diameter ratios the results are quite good.

Several problems which have also been solved as part of the requirements of undergraduate courses are presented in Fig. 8. Note that if the structures of Fig. 8 are above an infinite ground plane, image theory and the methods discussed here can be used to obtain solutions.

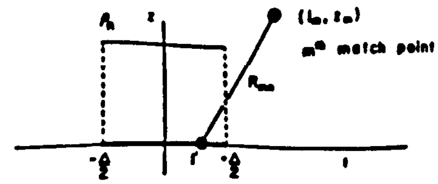

Fig. 6. Geometry for basic sub-problem for bent wire.

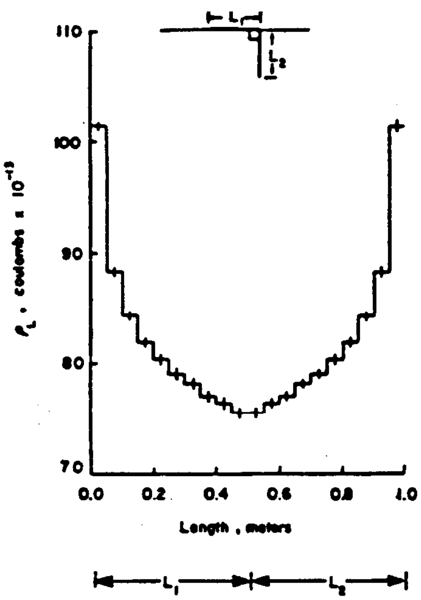

Fig. 7. Line charge density, ρ_L, versus length on a 1-meter wire bent in the middle with $\theta = 90°$ for $v = 1$ volt (20 unknowns). Wire radius = 1 mm.

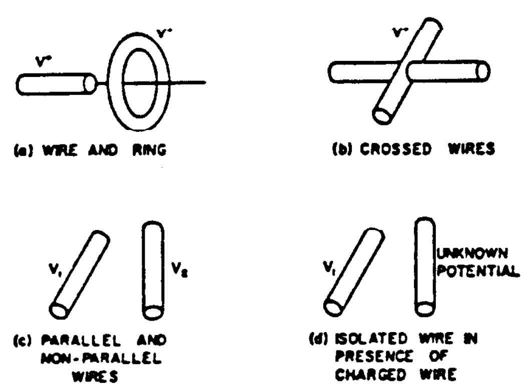

Fig. 8. Other wire-type problems solved in undergraduate courses at the University of Mississippi.

Two-Dimensional Example

The two-dimensional electrostatic boundary-value problem is of interest in many applications. For example, transmission

[2] Numerical integration can be used, but computation time will be increased.

line characteristics can be determined from "static" solutions of common TEM line configurations. For certain geometries, many of these problems have been solved analytically using surface equivalent models. However, numerical techniques can be used to extend to our knowledge to lines of arbitrary shapes found in many practical applications. To introduce the student to numerical solutions for such problems, one can consider the general two-dimensional TEM problem for contour (line) sources in a plane. In such cases, the scalar potential can be found from a potential integral similar to Eq. 1 as

$$\phi(y,z) = \frac{1}{2\pi\epsilon_0} \int_{\text{contour } c} \rho_c(y',z') \ln(R)\, dc. \tag{17}$$

In this equation, ρ_c is the total charge density, R is distance between a general field point and source point as defined in Eq. 12, and c is measured along the contour in the plane.

A specific example of this type problem is the two-dimensional, infinitely-thin strip problem shown in Fig. 9. If one again regards the integral for this problem as rectangular subareas in a numerical sense, then Eq. 17 becomes

$$\phi(y,z) = \frac{1}{2\pi\epsilon_0} \left[\sum_{n=1}^{M} \rho_n \int_{\Delta_{c_1}} \ln(R)\, dc + \sum_{m=M+1}^{N+M} \rho_n \int_{\Delta_{c_2}} \ln(R)\, dc \right] \tag{18}$$

where it is assumed that M subareas exist on contour c_1, N subareas exist on contour c_2, and the ρ_n's are the unknown amplitudes of the subareas. As before, Eq. 18 represents $N + M$ unknowns, and a set of $N + M$ linear equations can be obtained by considering $N + M$ equally-spaced match points on each strip or contour. Again, the scalar potential ϕ on each strip is assumed to be known. This set of equations will be of the same form as Eq. 8, where the individual coefficient terms derived from Eq. 18 will be of the form

$$l_{mn} = \int_{\Delta_n} \ln(R_{mn})\, dy'. \tag{19}$$

Here, R_{mn} is the distance from a point on the n^{th} source to the m^{th} field point, (Eq. 12), and Δ_n is the n^{th} subarea width as shown in Fig. 9c. Eq. 19 is integrable for the coefficient terms, l_{mn}, for the infinitely-thin strip problem even for the self-term (i.e., $m = n$). For simplicity subareas of width Δ are assumed in the basic subproblem of Fig. 9c. If the results of the integration of Eq. 19 for the l_{mn} terms are used along with the assumption that the potential on strip 1 is 1 volt and the potential on strip 2 is -1 volt, then a matrix equation of the form of Eq. 8 can be obtained in terms of the unknown charge densities, ρ_n's, and the V_n's terms of $2\pi\epsilon_0$.

This problem has been assigned to undergraduate students in junior and senior level courses, and the results of a computer solution of this system of equations using a matrix inversion routine, SIMQ[3], is shown in Fig. 10 for a number of segments per strip of $N = M = 3$ through 59. If one considers this as a strip transmission line, then the characteristic impedance can be determined from the computed charge densities as [6]

$$Z_0 = (\mu_0\epsilon_0)^{1/2}/C_l \tag{20}$$

where the capacitance per unit length, C_l, is

$$C_l = \frac{Q_c'}{V_d} = \sum_{n=1}^{M} \rho_n \Delta / V_d. \tag{21}$$

In these equations, Q_c is the total charge per unit length on one of the strips, V_d is the potential difference between the strips, and μ_0 and ϵ_0 relate to the characteristics of free space.

[3]SIMQ is a simultaneous linear equation solution routine from the IBM Scientific Subroutine Package.

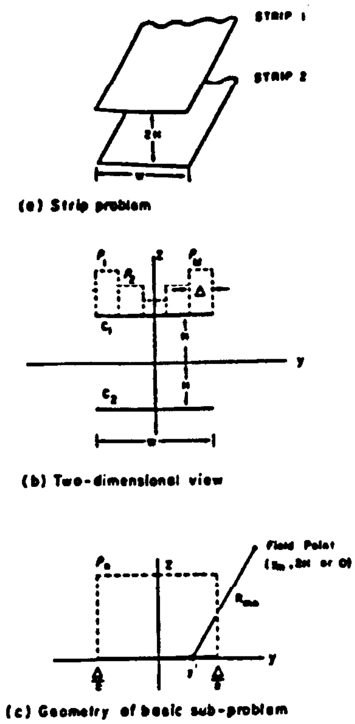

Fig. 9. Two-dimensional strip problem.

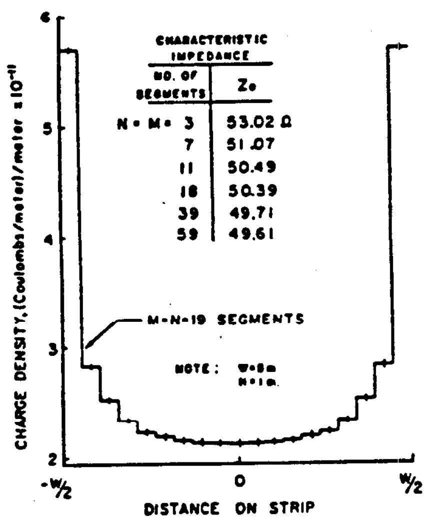

Fig. 10. Charge density versus distance for two strips with W/H = 5 (Z_0 ≈ 50 Ω from Wheeler's Curves [7]).

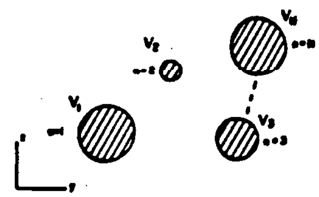

Fig. 11. Two-dimensional N-wire problem (See Reference [6] for solution).

For this example, a width-to-height ratio of 5 was selected for a Z_0 of approximately 50 ohms based on Wheeler's impedance curves for a relative dielectric constant of 1 [7]. The Z_0's of this line have been computed numerically for several numbers of segments on each strip to show the trend in convergence which are tabulated in Fig. 10 (see insert).

There are many other important two-dimensional configurations that can be treated with the approach outlined. One important case is the N-wire problem of Fig. 11 which has been analyzed by Clements, Paul, and Adams in their study of dielectric-coated circular wires [8].

Time-Harmonic Example

For the average student, it is difficult to learn several new concepts simultaneously. Therefore, the approach to moment methods for undergraduates has been limited to (1) the interpretation of an integral as a sum of a finite number of subareas as presented in basic calculus courses and (2) the concepts of linear systems of equations which have already been encountered in electric circuits courses. With this approach, the student is simply asked to solve new problems with a digital computer based on previously attained knowledge.

When the student acquires advanced knowledge, more of the concepts of moment methods can be introduced as part of a course without detracting from the normal presentation of course material. Thus, as the student obtains a better understanding of the representation and application of functions with series expansions from the study of Fourier series in signal analysis, one can easily introduce the concept of expansion functions or basis functions to obtain moment method-type solutions for field problems [1].

The extension of the integral representation approach used previously to the idea of pulse expansion functions with point matching is obvious [1], and this approach has been used with success after one or two problems have been solved with the integral approach. The use of entire domain expansion functions can easily be related to the Fourier series representation of functions, a concept that is well-known to students.

In senior-level fields and design courses, we have undertaken the numerical solution of a Hallén integral equation for the current distribution on a thin, center-fed, half-wave length dipole antenna as shown in Fig. 12. In this analysis, a truncated series expansion proposed by Neff, Siller, and Tillman [9] consisting of functions defined over the total length of the antenna of the form,

$$I_z(z) = \sum_{n=1}^{2} B_n \sin\left(n\pi/2H(H - |z|)\right), \tag{22}$$

is used where $H = \lambda/4$.

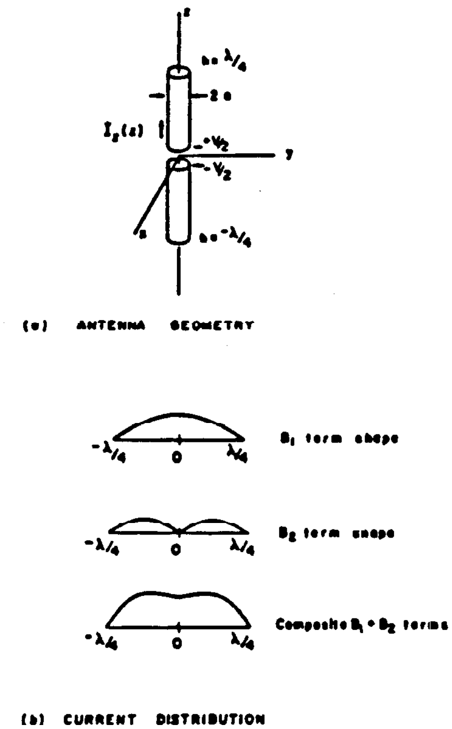

Fig. 12. Thin cylindrical center-fed dipole antenna with radius a and length $\lambda/2$.

In Fig. 12b, a very simple composite or sum of the two term series for the current of Eq. 22 is presented to show that the approximate current does indeed "add up to" or resemble the basic shape of the current distribution measured by Mack on a half-wave dipole antenna [10]. Hence, for the analysis of this problem the approximate series representation for the current of Eq. 22 is substituted into the Hallén integral equation,

$$\int_{-\lambda/4}^{\lambda/4} I_z(z')K(a,z,z')\,dz' = -j(4\pi/\eta)(C_1 \cos k_0 z + V/2 \sin k_0|z|) \tag{23}$$

where η is the impedance of the free space, V is the potential of the delta function excitation, k_0 is the free-space propagation constant and $I_z(z')$ is the unknown current distribution. In this case, the approximate kernal of the integral equation is assumed to be

$$K(a,z,z') = \frac{\exp\{-jk_0[(z-z')^2 + a^2]^{1/2}\}}{[(z-z')^2 + a^2]^{1/2}} \tag{24}$$

where a is the radius, z is an observation or field point, and z' is a source point. The Hallén formulation, which has been investigated extensively in the literature, is normally derived in a simplified form in the class presentation as a boundary value problem. Now, if Eq. 22, which represents the current distribution, is substituted into Eq. 23, one equation in terms of three unknowns is obtained, i.e.

$$\sum_{n=1}^{2} B_n \int_{-\lambda/4}^{\lambda/4} \sin\left(\frac{2\pi n}{\lambda}(\lambda/4 - |z'|)\right) K(a,z,z')\, dz' + \frac{j4\pi}{\eta} C_1 \cos k_0 z = -\frac{j2\pi V}{\eta} \sin k_0 |z| \qquad (25)$$

where B_1, B_2, and C_1 are the unknowns. Three equations in terms of the three unknowns can be obtained as before by evaluating Eq. 25 at three points in z, i.e., point matching. These points are normally equally spaced to obtain independent equations, such as $z = 0, \lambda/8, \lambda/4$. A matrix equation of the form of Eq. 8 can now be generated from the l_{mn} coefficient functions

$$j\frac{4\pi}{\eta} \cos k_0 z_m \qquad (26)$$

and

$$\int_{-\lambda/4}^{\lambda/4} \sin\left(\frac{2\pi n}{\lambda}(\lambda/4 - |z_m|)\right) K(a, z_m, z')\, dz'; \qquad n = 1, 2 \qquad (27)$$

where m is the m^{th} match point. Here, n is the n^{th} term of the series expansion for the current which is a source of specified distribution over the entire length of the antenna. In this context, the index, n signifies the "harmonic" of the sinusoidal expansion function used in the approximation. Evaluation of the coefficients of the linear system of equation describing this problem are now complex numbers for the time-harmonic problem rather than real numbers as in static problems which complicates the programming somewhat.

It is well to note that the evaluation of the integral in Eq. 27 can introduce errors in the solution if it is not treated carefully. However, standard integration routines for complex functions have been used to obtain these coefficients with very good results if an appropriate number of integration points are used. The integrand is a very "peaked" function as a result of the "delta function nature" of the kernal function when $z' = z_m$ for a small radius, and it is easy to introduce large errors in computing this integral if a major contribution of the integrand is omitted because of poor selection of the number and placement of integration points [11].

The simple 3 X 3 matrix equation, which can be generated from the evaluation of Eqs. 25, 26, and 27 for this problem, is readily solved for B_1, B_2, and C_1 using matrix solution routines for complex numbers. For the half-wavelength antenna with radius $a/\lambda = 7.022 \times 10^{-3}$, the related matrix equation becomes (note form; (real, imaginary))

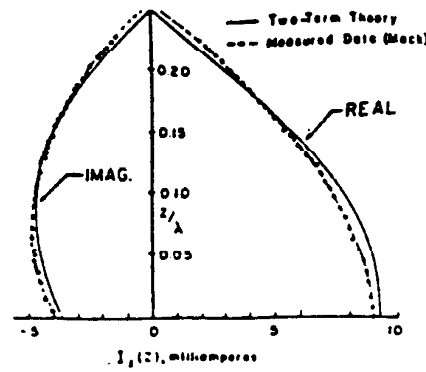

Fig. 13. Current on the upper half of a half wave dipole antenna for $a/\lambda = 7.022 \times 10^{-3}$

where $V = 1$ volt. Subsequent solution of Eq. 28 gives the values of $B_1 = (0.00940354, -0.00357376)$ and $B_2 = (0.00045342, -0.00203125)$ as the complex coefficients of the series expansion. A plot of the current distribution as a function of z computed from the series representation of Eq. 22 with the computed values of B_1 and B_2 is presented in Fig. 13 along with Mack's measured data [10]. These results are quite accurate when compared with actual measurements of current distribution which further emphasizes the educational benefits to the student of solving a "real-world" fields problem from a mathematical model.

This approach, as well as pulse-expansion and point matching, has been extended by senior students to solve time-harmonic problems of general arrays of parallel wire antennas, wire antennas with electrically-small top loading, and other problems related to wire antenna structures with parallel currents. These type problems are much more complex than any of the previous problems, and, therefore, undergraduate students normally are assigned such advanced problems only as a project in a two semester hour design course.

IV. Conclusions

The thesis of this paper on teaching the undergraduate student numerical techniques for the analysis of electromagnetic problems is based on the simple and direct procedure for solving integral equations [6] which is commonly known as point matching or collocation [12]. In fact, this solution procedure is not formulated in terms of moment methods. However, the proposed treatment of such problems is a special case of the method of moments. It readily follows that the related functional equations can be reduced to matrix equations similar to those derived in the previous sections of this paper through the formal application of moment methods.

The emphasis in the presentation of the "point matching" approach is on obtaining a solution for a "real-world" fields problem of practical geometry based upon the student's existing mathematical background rather than on the unified theory of the solution of fields problems by moment methods.

$$\begin{bmatrix} (6.88297898,-1.85131747) & (2.80690163,-1.76841739) & (0.0,0.03333333) \\ (4.89364523,-1.67814235) & (5.95102084,-1.60983890) & (0.0,0.02357023) \\ (0.66511328,-1.21834007) & (1.12957917,-1.18771271) & (0.0,0.00000000) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0.0,0.00000000 \\ 0.0,-0.01178511 \\ 0.0,-0.01666667 \end{bmatrix} \qquad (28)$$

If solutions are obtained from such problems as outlined in the results section, they should serve to motivate the student to learn more about the rather "abstract" area of field theory while providing the soon-to-be graduate engineer with a versatile tool for the solution of practical problems.

Student response to this departure from the traditional presentation of analysis of fields problems has been excellent as described earlier. We are continuing to evaluate the impact of this technology on our educational program and its effect on our practicing graduates. This has led to the compilation of the examples that have been used in the classroom to (1) document the results of an educational experiment on teaching numerical methods in electromagnetics to undergraduates and to (2) provide the educator involved in fields courses a simplified approach (or guide) to teaching moment methods to undergraduate students.

In conclusion, it should be added that the integration of this new instruction technique in any program should be tempered by a comment made earlier by Schelkunoff on teaching electromagnetics where he indicated that "we should introduce new physical and mathematical concepts gradually, as the occasion demands, and start using new mathematical techniques when it becomes obvious that they are really needed" [13].

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