Types: Fredholm

1st Kind: $f(x) = \int_a^b \underbrace{K(x,t)}_{\text{Kernel (Green's } F^n)} \underbrace{\Phi(t)}_{\text{unknown}} \, dt$ ← $x'$

2nd Kind: $f(x) = \Phi(x) - \lambda \int_a^b K(x,t)\, \Phi(t)\, dt$

↑ constant (real or complex)

3rd kind $f(x) = a(x)\, \Phi(x) - \lambda \int_a^b K(x,t)\, \Phi(t)\, dt$

Types: Volterra

1st Kind: $f(x) = \int_a^x K(x,t)\, \Phi(t)\, dt$

2nd Kind: $f(x) = \Phi(x) - \lambda \int_a^x K(x,t)\, \Phi(t)\, dt$

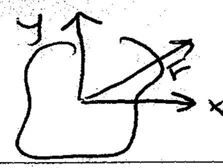
3rd Kind: $f(x) = a(x)\, \Phi(x) - \lambda \int_a^x K(x,t)\, \Phi(t)\, dt$

Examples:

Statics $V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int_{v'} \underbrace{\rho_v(\bar{r}')}_{\Phi(r')} \underbrace{\frac{1}{|\bar{r}-\bar{r}'|}}_{K(\bar{r},\bar{r}')} \, dr'$

integrate over all charges; charge density (C/m³); test location; source location

Fredholm 1st Kind $\int_a^b \Phi(r')\; K(\bar{r},\bar{r}')\, dr'$




$\beta = 2\pi/\lambda$ test loc. page 2 source loc.

Scattering 2D

$$E_z(\bar{r}) = \frac{-k\eta_0}{4} \int_s \underbrace{J_z(\bar{r}')}_{\substack{\text{surface} \\ \text{current density} \\ A/m^2 \\ \Phi(\bar{r}')}} \underbrace{H_0^{(2)}(k|\bar{r}-\bar{r}'|)}_{\substack{\text{Henkel } f^{n} \text{ of } 2^{nd} \\ \text{kind} \\ K(\bar{r},\bar{r}')}} \, dS'$$

$k$: $\beta$

↑ z directed electric field

Test point

Radiation from antenna

↑ $I(z)$ small current $\quad G(\bar{r},\bar{r}') = \dfrac{e^{-jkR}}{4\pi R}$

$R = |\bar{r}-\bar{r}'|$

$E_z(\bar{r})$ from $I(z) = 1A$

$$E_z = -j\omega\mu \left(1 + k^2 \frac{\partial^2}{\partial z^2}\right) \int_{-l/2}^{l/2} I(z')\, G(z,z')\, dz'$$

↑ not easy to evaluate

Pocklington's Eq$^{n}$ brings $\partial/\partial z$ inside integral

$$j\omega\varepsilon E_z = \int_{l/2}^{l/2} I(z') \left(\frac{\partial^2}{\partial z^2} + k^2\right) \frac{e^{-jkR}}{R}\, dz'$$

$R = |z - z'|$

or $R = |(x-x')^2 + (y-y')^2 + (z-z')^2|^{1/2}$

Hallen's uses analytical theory to simplify eq$^{n}$

$$\underbrace{k^2 \left(\frac{-E_z}{j\omega\mu}\right)}_{S(z)} = \left(\frac{\partial^2}{\partial z^2} + k^2\right) \underbrace{\left(\int_{-l/2}^{l/2} I(z')\, G(z,z')\, dz'\right)}_{F(z)}$$

Solve this for homogeneous & particular sol$^{\text{ns}}$:

Homogeneous

$$\left(\frac{\partial^2}{\partial z^2} + \underset{\beta^2}{k^2}\right) F(z) = 0$$

$$F_h(z) = c_1 \cos kz + c_2 \sin kz$$

Particular:

$$F_p(z) = k \int_{-\ell/2}^{\ell/2} S(z') \sin k|z - z'| \, dz'$$

Add

$$\int_{-\ell/2}^{\ell/2} I(z') G(z, z') \, dz' = c_1 \cos kz + c_2 \sin kz - \frac{j}{\underset{\sqrt{\mu/\varepsilon}}{\eta}} \int_{-\ell/2}^{\ell/2} E_z(z') \sin k|z - z'| \, dz'$$

Where do Volterra Integrals come from?

Integral-Differential Eq$^{\text{n}}$ transformation

Solve $\dfrac{d\phi}{dx} = F(x, \phi) \quad a \le x \le b$ for $\phi(a) =$ constant (boundary condition)

$$\phi(x) = \int_a^x F(t, \phi(t)) \, dt + \underset{\phi(a)}{c_1}$$

$$= \phi(a) + \int_a^x F(t, \phi) \, dt \qquad \text{Volterra } 1^{\text{st}} \text{ kind}$$

Now solve $\dfrac{d^2\phi}{dx^2} = F(x, \phi) \quad a \le x \le b$

$$\frac{d\phi}{dx} = \int_a^x F(x, \phi(t)) \, dt + c_1$$

$$\phi(x) = \underset{\phi(a)}{c_2} + \underset{\phi'(a)}{c_1} \overset{(x-a)}{x} + \int_a^x (x - t) F(x, \phi(t)) \, dt \qquad \text{Volterra } 2^{\text{nd}} \text{ Kind}$$