ECE 6340 Finite Element Method (FEM)

Maxwell's Equations - Point Form (Differential Form) Time Domain
^^^ Used for Discrete (numerical) solution

$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

$\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\text{source}$

$\nabla \bullet \mathbf{B} = 0$

$\nabla \bullet \mathbf{D} = q_v$ source

Constituitive Relationships

Assuming "Simple Media"

- Uniform in space (at least the small discrete space where you are working)
- Single Frequency or Frequency Independent Properties
- NOT time varying

$\mathbf{D} = \varepsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

Units

$\mathbf{E}$ = V/m = N/C Electric Field
$\mathbf{D} = C/m^2$ Electric Flux Density
$\mathbf{H}$ = A/m Magnetic Field
$\mathbf{B} = Wb/m^2$ = Tesla = N/(A*m) Magnetic Flux Density
$q_v = C/m^3$ Charge Density (volume)
$\mathbf{J}$ = A/m Current Density
$\nabla = \partial \mathbf{x} / \partial x + \partial \mathbf{y} / \partial y + \partial \mathbf{z} / \partial z$

Maxwell's Equations: Static Form

Substitute $\partial/\partial t = 0$ in above equations.

$\nabla \times \mathbf{E} = 0$ (1)

$\nabla \times \mathbf{H} = \mathbf{J}$

$\nabla \bullet \mathbf{H} = 0$

$\nabla \bullet \mathbf{E} = q_v / \varepsilon$ (2)

Define Potential = Φ (volts)

$\mathbf{E} = -\nabla \Phi$

Substitute into (1) and (2) above.

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$\nabla \times (-\nabla\Phi) = 0$ which satisfies the vector identity: Del x DelA = 0

$\nabla * (-\nabla \Phi) = q_v/\varepsilon$

Which gives:

Poisson's Equation:

$$\nabla^2 \Phi = - q_v / \varepsilon$$

Laplace's Equation (Poisson's for a source-free region)

$$\nabla^2 \Phi = 0$$

Maxwell's Equations (Dynamic Forms)

$q_v = 0$ (no free charges floating around)

a) TRANSIENT:

NO further assumptions. Solve the above Maxwell's Equations in the time domain.

b) STEADY STATE:

Assume sinusoidal fields: $\mathbf{E}(t) = \mathbf{E}o\, e^{j\omega t}$

$$\frac{\partial \mathbf{E}(t)}{\partial t} = j\omega \mathbf{E}o\, e^{j\omega t} = j\omega\, \mathbf{E}(t)$$

Assume Simple Media: $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (1)$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{Jsource} \quad (2)$$

$$\nabla \bullet \mathbf{H} = 0$$

$$\nabla \bullet \mathbf{E} = 0$$

Solve for $\mathbf{H}$ in (1): $\mathbf{H} = \dfrac{-1}{j\omega\mu} \nabla \times \mathbf{E}$

Plug into (2): $\dfrac{-1}{j\omega\mu} \nabla \times \nabla \times \mathbf{E} = j\omega\varepsilon\mathbf{E} + \mathbf{J}\text{source}$

Use vector identity: $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E}$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \wedge\wedge = 0$$

Gives Vector Wave Equation

$$(\nabla^2 - \omega^2 \mu\varepsilon)\, \mathbf{E} = j\omega\mu\, \mathbf{J}\text{source}$$

wavenumber: $k = \omega \sqrt{(\mu\varepsilon)} = 2\pi / \lambda_\varepsilon$

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wavelength: $\lambda_\varepsilon = c_\varepsilon / f = 2 \pi c_\varepsilon / \omega$
speed of propagation : $c_\varepsilon = c_o / \sqrt{(\varepsilon_r)} = 1/\sqrt{(\mu_o \varepsilon_o \varepsilon_r)} = 1/\sqrt{(\mu\varepsilon)}$

This gives the <u>Inhomogenous vector wave equation:</u>

$(\nabla^2 - k^2) \mathbf{E} = j\omega\mu \mathbf{J}\text{source}$

For a source-free region, we have the <u>Homogeneous vector wave equation</u>

$(\nabla^2 - k^2) \mathbf{E} = 0$

For the homogeneous case, the vector equation can be divided into independent x,y,z components, so becomes a <u>scalar wave equation</u>

$(\nabla^2 - k^2) E_x = 0$
$(\nabla^2 - k^2) E_y = 0$
$(\nabla^2 - k^2) E_z = 0$

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