ECE 5340/6340
FINITE-DIFFERENCE FREQUENCY-DOMAIN METHOD

- (FD-FD or FD method)
- Used to solve ordinary partial differential equations (ODEs)

$$Laplace's Equation: \nabla^2 \Phi = 0$$

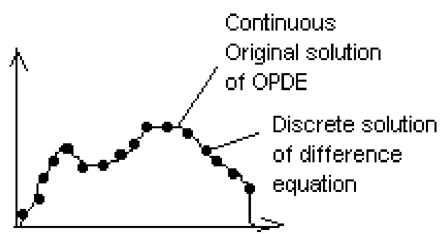
$$Poisson's Equation: \nabla^2 \Phi = -q_v/\varepsilon$$

$$Helmholtz Equation: (\nabla^2 + k_\varepsilon^2)\Phi = 0$$

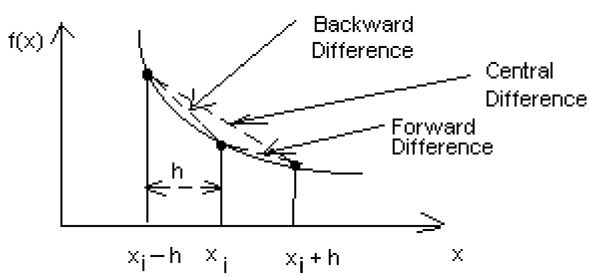
Laplace's Equation used for electrostatics, TM lines
Poisson's Equation used for semiconductors
Helmholtz Equation used for waveguides

FDTD METHOD:

Differential Equation → Difference Equation → System of linear equations → Solve → Values at discrete points



NUMERICAL DIFFERENTIATION



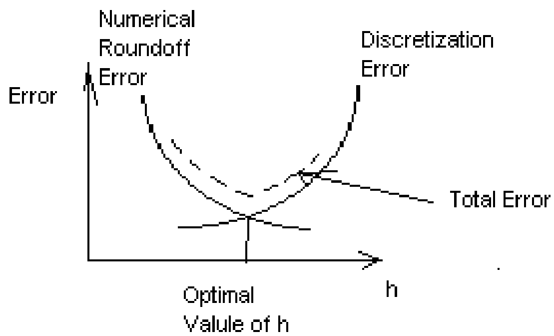
$$\frac{\partial f}{\partial x} = f' = slope\_of\_curve$$

$$f_i' \cong \frac{f_{i+1} - f_i}{h} Forward\_Difference$$

$$f_i' \cong \frac{f_i - f_i - 1}{h} Backward\_Difference$$

$$f_i' \cong \frac{f_{i+1} - f_{i-1}}{2h} Central\_Difference$$

TYPES OF ERROR:



a) Discretization Error: h is too large to represent derivative
b) Round-off Error: h is too small. Difference (a-b) is so small that computer has roundoff error.

Optimal value of h is approx. wavelength/10 to wavelength/50
For most simulations.

ERROR ANALYSIS of NUMERICAL DIFFERENTIATION
(Digitization error):
Taylor expansion:

(1) $f_{i+1} = f_i + h f_i' + (h^2/2) f_i'' + \dots$ Forward Taylor Series

$f_i' = \frac{f_{i+1} - f_i}{h} - (h/2) f_i'' + \dots$ Forward Difference Equation
^^Error on the order of: $(h/2) f_i''$ (overestimate)

(2) $f_{i-1} = f_i - h f_i' + (h^2/2) f_i'' - (h^3/6) f_i''' + \dots$ Backward Taylor Series

$f_i' = \frac{f_i - f_{i-1}}{h} + (h/2) f_i'' + \dots$ Backward Difference Equation
^^Error on the order of: $(h/2) f_i''$ (underestimate)

Take (1) - (2):
$f_{i+1} - f_{i-1} = 2h f_i' + (h^3/3) f_i'' + \dots$

$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} - (h^2/6)\, f_i'''$ Central Difference Equation

^^Error on the order of: $(h^2/6)\, f_i'''$ (underestimate)

EXAMPLE:

$f(x) = x^2$ $f'(x) = 2x$ $f''(x) = 2$

$f(x) = [4\ \ 9\ 16]$

$x\ \ = [2\ \ 3\ \ 4]$

$h = 0.5$ because $f_{i+1} = 3$ and $f_{i-1} = 2$ $(2h = 3-2)$

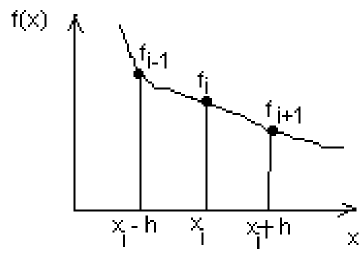
$f'(x=2.5) = (9-4)/1 = 5$

$f'(x=3.5) = (16-9)/1 = 7$

Error $= h^2 f''' / 6 = 0$ (predicted)

$= 0$ (observed)

2^{nd} DERIVATIVE (which is required by ∇^2)



$f'(x_i + h/2) = (f_{i+1} - f_i) / h$ + error $((h^2/ 6)\ f''')$

$f'(x_i - h/2) = (f_i - f_{i-1}) / h$ + error $(\ (h^2/ 6\ )\ f''')$

$f''(x_i) = (f'(x_i + h/2) - f'(x_i - h/2)) / h$

$= (f_{i+1} - 2f_i + f_{i-1}\ ) / h^2$ + error $((h^2/ 3\ )\ f''')$

Note: These error terms are ORDER OF THE ERROR, because all higher-order (higher orders of h) error terms were neglected.

Example (continued)

$h = 1$ in this case, because $f_{i+1} = 4$ and $f_{\ i-1} = 2$ $(2h = 4-2)$

$f''(x=3) = (f'(x=3.5) - f'(x=2.5))/ (2*1) = (7 - 5) / 1 =2$

OR: $= (16 - 2(9) + 4) / (1^2) = 2$

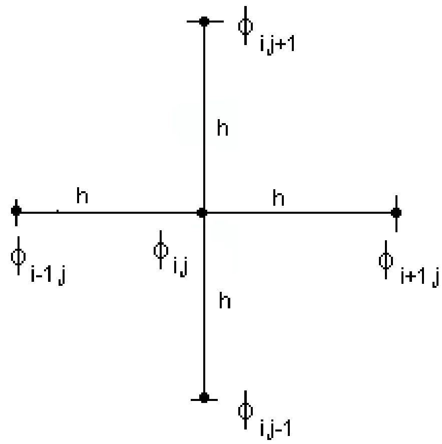
Error $= 0$ because $f''' = 0$

FDFD METHOD:

Application of numerical differentiation to 2D Laplace Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

5-point stencil commonly used for FDFD:



$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

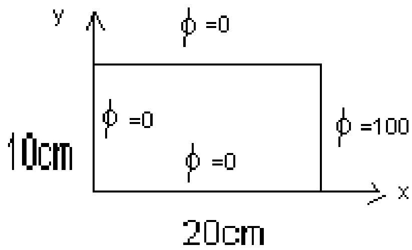
$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{h^2}$$

$$\nabla^2 \phi = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{h^2}$$

This is the 5-point Difference Operator for Equi-spaced Data (hx=hy=h) … Equal-Arm Star, or Uniform 5-point stencil

Example 1: Rectangular Duct
Find the potential distribution.



Solution should satisfy Laplace's Equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Subject to the BOUNDARY CONDITIONS:

$$\phi(x,0) = \phi(x,10) = \phi(0,y) = 0$$

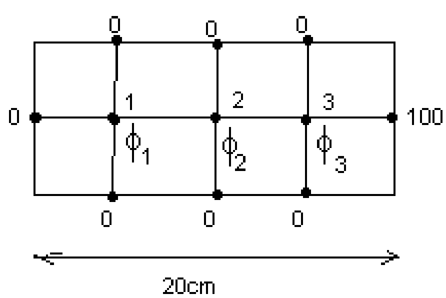
$$\phi(20,y) = 100 volts$$

1. Replace differential equation by difference operator

$$\nabla^2 \phi = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{h^2}$$

where φij are discrete values of potential at points (nodes) within domain of interest

2. Lay out coarse mesh



h=20 cm / 4 divisions = 5 cm = 0.05 meters

3. Apply Difference Equation at Each Node (1,2,3)

Node 1: [0+0+0+P2 - 4*P1] / 0.05 ^{2} = 0

Node 2: [P1+0+0+P3-4*P2]/ 0.05 ^{2} = 0

Node 3: [P2+100+0+0-4*P3]/ 0.05 ^{2} = 0

(Multiply through by 0.05^2)

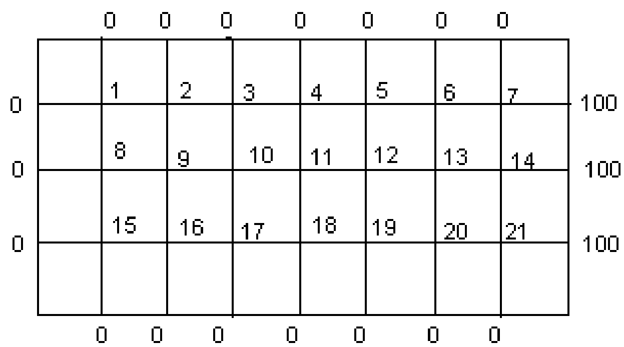
4. Solve:

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -100 \end{bmatrix}$$

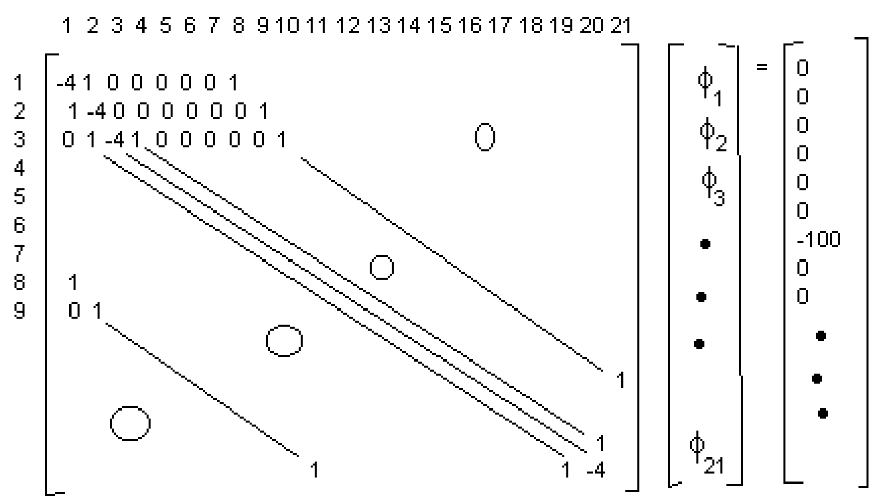
Gives:

P1 = 1.786, P2 = 7.143, P3 = 26.786

5. Repeat for Refined Mesh: h=2.5 cm



6. Again write 21 equations in 21 unknowns



Standard FDFD matrix for equal-arm star and Laplace's equation: a -4 appears on diagonal, and a "1" appears for each neighboring element. This is a banded matrix, 21x21.
Array of unknowns, 21x1
b vector: represents boundary conditions … -100 touches P7, P14, P21. All others are 0.

7. Solve:

	h=5cm		h=2.5cm		Analyt.
	Value	Err	Value	Err	Value
P1	1.786	-.692	1.289	-.195	1.0943
P2	7.143	-1.655	6.019	-.531	5.4885
P3	26.786	-.692	26.289	-.195	26.0944

Conclusion:

FDFD operator can be used to solve Laplace, Poisson, Helmholtz equations.

Error is caused by linear approximation to derivative and is on the order of $h^2 f'''$ for equal-spaced points. Unequal-spaced points can be used, but the error will then be on the order of hf''.

FDFD results in a banded matrix with "4"s on the diagonal, "1" on neighboring elements. The "b" matrix has zeros except where non-zero boundary conditions exist.