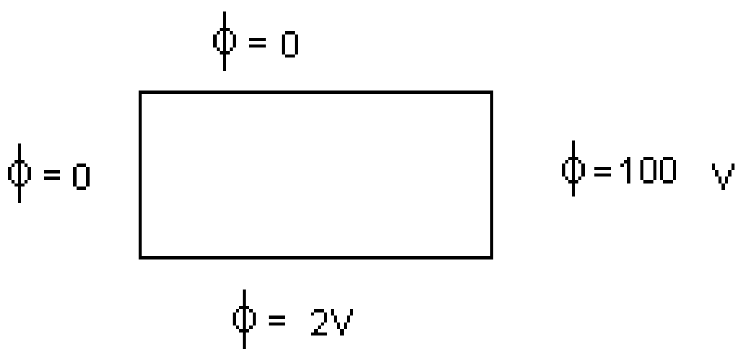
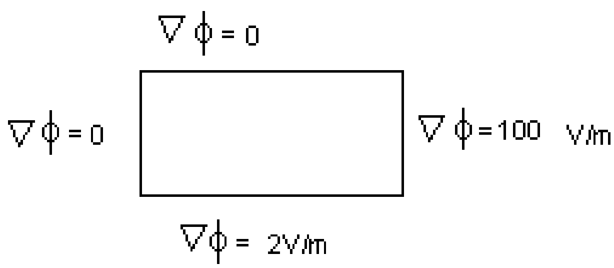
ECE 5340/6340

FINITE DIFFERENCE FREQUENCY DOMAIN
BOUNDARY CONDITIONS

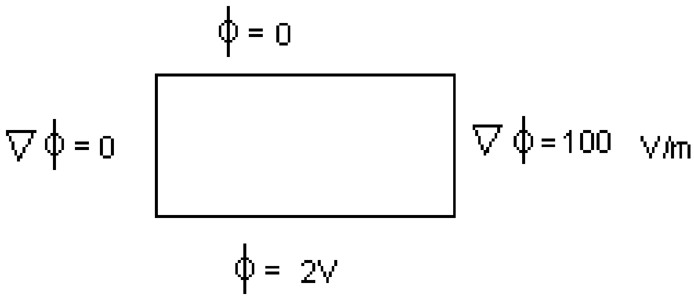
Dirichlet: $\phi_{I,j}$ is specified at the boundary
We saw how to handle these in the last lecture …
set $\phi_{I,j}$ to whatever constant is applicable on the external nodes.
Corners do not matter, because they are not used.



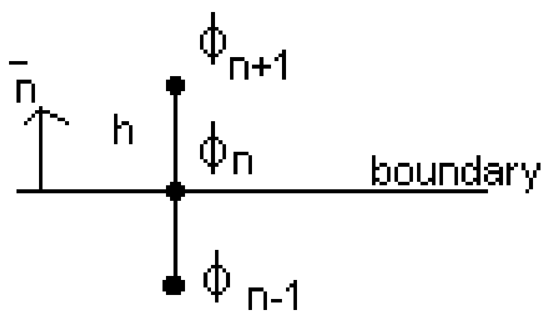
Neuman: $\nabla\phi$ is specified at the boundary



Combination:



Neumann Boundary Conditions:
$\nabla\phi$ is specified at the boundary



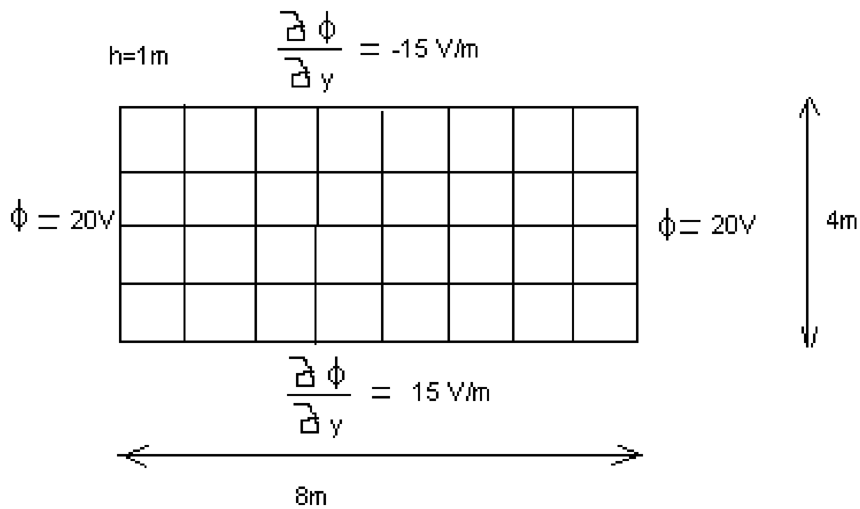
Given $\partial\phi/\partial y = -15$ V/m on boundary:

$\partial\phi/\partial y = (\phi_{n+1} - \phi_{n-1})/(2h) = -15$ V/m (Central difference formula)

$\phi_{n+1} = (-15 \text{ V/m})\, 2h + \phi_{n-1}$

<u>To Program this:</u> ADD an extra row of points outside of the boundary (ϕ_{n+1}), and solve for them using the boundary condition equation above.

EXAMPLE 3: Solving Poisson's Equation with Mixed Boundary Conditions and Symmetry



Solve Poisson's Equation: $\nabla^2\phi = -q_v/\varepsilon$

$[\phi_{I-1,j} + \phi_{I+1,j} + \phi_{I,j-1} + \phi_{I,j+1} - 4\,\phi_{I,j}] = -q_v h^2/\varepsilon$

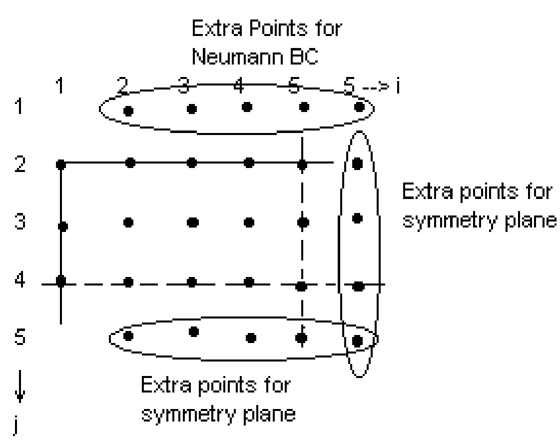
Using SOR:

$$\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + \omega R_{i,j}^{(k)}$$

$$R_{i,j}^{(k)} = \frac{\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k)} + \phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k)} - 4\phi_{i,j}^{(k)} - b_{i,j}}{4}$$

$$b_{i,j} = -h^2 q_v / \varepsilon$$

Discretize and Apply Boundary Conditions and Symmetry:



OBSERVATION:
Symmetry boundary is equivalent to a Neumann boundary condition with $\partial\phi/\partial y = 0$

Boundary conditions drive an FDFD problem. If all of the boundary conditions are zero, then the solution will also be zero.

It is also worth noting that boundary conditions do not have to be applied on an outside boundary. They can also be applied internal to the object, wherever they are physically applicable.

Programming Examples (see website):
See handouts on this problem. One does it using a matrix (standard FDFD form), and the other does it using SOR (incl. Program).

DO BOUNDARY CONDITIONS PROVIDE A UNIQUE SOLUTION?

Proof: Postulate two different (non-unique) solutions and find conditions necessary to make them identical (unique)

Assume two possible solutions:
$\nabla^2\phi_1 = -q_v/\varepsilon$ $\qquad\qquad$ $\nabla^2\phi_2 = -q_v/\varepsilon$ (1)

Suppose ϕ_1 and ϕ_2 also satisfy the boundary conditions.
From (1) there will be a difference potential.

$$\phi_o = \phi_1 - \phi_2$$

which satisfies Laplace's Equation (linear equation, can use superposition) $\nabla^2\phi = 0$

Vector identity: $\nabla\bullet(\phi_o\nabla\phi_o)=\phi_o\nabla^2\phi_o+\phi_o\bullet\nabla\phi_o$

^^=0 from Laplace's equation

Now integrate over volume V and apply Divergence Thm:

$$\int_v (\overline{\nabla} \bullet \overline{F}) d\overline{V} = \oint_S \overline{F} \bullet \overline{dS}$$

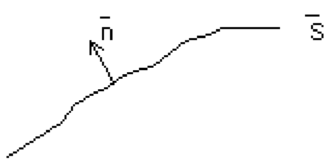
$$\int_V \nabla \bullet (\phi_o \nabla \phi_o) dv = \oint_S \phi_o \nabla \phi \bullet dS = \int_V \nabla \phi_o \bullet \nabla \phi_o dV = \int_V |\nabla \phi_o|^2 \, dV$$

$$where..\phi_o \nabla \phi_o = F$$

$U \sin g$:

$$\oint_S \phi_o \nabla \phi \bullet dS = \int_V |\nabla \phi_o|^2 \, dV$$

And realizing that: $\nabla\phi \bullet dS = \partial\phi/\partial n$ is the component of the gradient of ϕ normal to S:



$$\oint_S \phi_o \frac{\partial \phi_o}{\partial n} dS = \int_V |\nabla \phi_o|^2 dV$$

To have a unique solution for $E = -\nabla \phi_o$
requires $\phi_1 = \phi_2$, so $\nabla\phi_1 = \nabla\phi_2$, or $\nabla \phi_o = 0$

This makes the RH term of the above equation =0.
The LH term will =0

IF : Dirichlet BC: $\phi_o = 0$ >> ϕ_i are specified for all points on BC

ELSEIF: Neumann BC: $\nabla \phi_o = 0$ >> $\partial\phi i/\partial n = 0$ are specified for all points on BC.