Field Theory for Transmission Lines

Pozar Chapter 2.2

Objective: Calculate RLGC parameters given only transmission line geometry

Reference Material (contains info on many types of lines that are not in our text!): Transmission Line Handbook

Start with L&C:

From the definition of stored magnetic energy density: (Reference: Ulaby Sect 5-9)

$$W_m = \frac{1}{2} \underbrace{L}_{\text{inductance}} I^2 = \frac{1}{2}\mu H^2 \quad (J/m^3)$$

and electric energy density: (Ref Ulaby 4-11)

$$W_e = \frac{1}{2} \underbrace{C}_{\text{capacitance}} V^2 = \frac{1}{2}\varepsilon E^2 \quad (J/m^3)$$

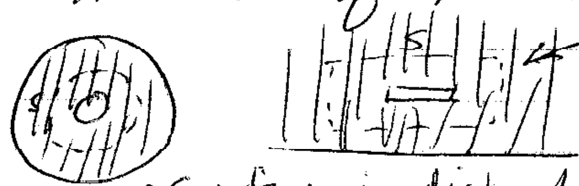
If we know E, we can find C, and if we know H, we can find L

Converting from energy density (J/m^3) to time averaged energy (J):

$$W_m = \frac{\mu}{2} \int_S \frac{\bar{H}}{\sqrt{2}} \cdot \frac{\bar{H}^*}{\sqrt{2}} \, dS \qquad (J)$$

convert from $\bar{H}$ peak to RMS for time averaged energy

where S is a closed surface around the Xsection of the line



surface

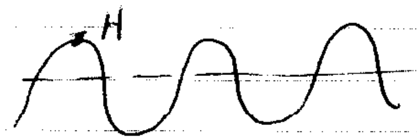
Surface is disk w/ hole cut out

$$W_e = \frac{\varepsilon}{2} \int_S \frac{\bar{E}}{\sqrt{2}} \cdot \frac{\bar{E}^*}{\sqrt{2}} \, dS \qquad (J)$$

So $$L = \frac{\mu}{|I_0|^2} \int_S \bar{H} \cdot \bar{H}^* \, dS \qquad (H/m)$$

peak $\bar{H}$, magnitude of $\bar{H}$

per unit length



$$C = \frac{\varepsilon}{|V_0|^2} \int_S \bar{E} \cdot \bar{E}^* \, dS \qquad (F/m)$$

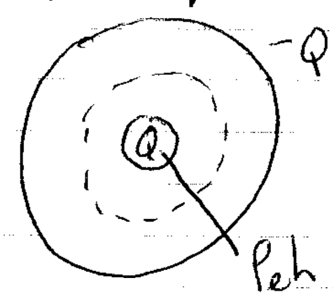
How do we find $\bar{E}, \bar{H}$?

Several ways...

(1) Solve Laplace's Eq^{n} subject to boundary condition $E_{tangential} = 0$ on metal. Can be done analytically (take Advanced EM course) or numerically (take Numerical EM course)

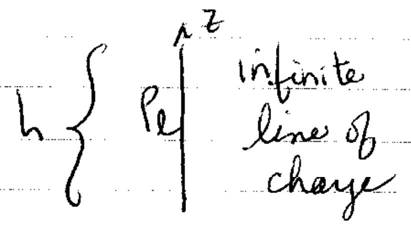
(2) See Ulaby for another method...

Example for a coaxial line



Using Gauss Law for E (Ulaby Sect 4.4)

$$\oint_S \hat{D} \cdot d\bar{S} = Q_{enclosed}$$



$$\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{r} D_r \cdot \hat{r}\, r\, d\phi\, dz = \rho_\ell h$$

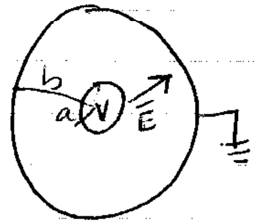
$$2\pi r h D_r = \rho_\ell h$$

$$\bar{D} = \frac{\rho_\ell}{2\pi r} \hat{r} = \varepsilon \bar{E}$$

$$\bar{D} = \frac{Q}{2\pi r h} \hat{r} = \varepsilon \bar{E}$$

Let's relate $\bar{E}$ to V instead of Q to enable calculation of C

$$V = +\int_b \bar{E}\cdot d\bar{\ell}$$



$$= +\int_a^b E_r \hat{r}\cdot\hat{r}\,dr$$

$$= +\int_a^b \frac{Q}{2\pi\varepsilon h}\,\hat{r}\circ\hat{r}\,dr = \frac{Q}{2\pi\varepsilon h}\ln\left(\frac{b}{a}\right)$$

$$\bar{E} = \frac{V}{r\ln(b/a)}\,\hat{r}$$

Include traveling wave parameter

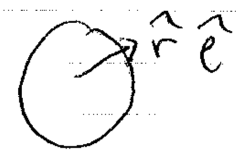
$$\bar{E} = \frac{V}{r\ln(b/a)}\,\hat{r}\,e^{-\gamma z}$$

Pozar starts here

$$\gamma = \alpha + j\beta$$

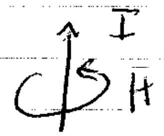
Note Ulaby uses $\hat{r}$ for

Pozar uses $\hat{\rho}$



Now find $\overline{H}$:

infinite line of current (Ulaby sect 5-4.2)



$$\oint_C \overline{H} \cdot d\overline{l} = I \qquad \text{Ampere's Law}$$

$$\int_{\phi=0}^{2\pi} H_\phi \hat{\phi} \cdot \hat{\phi}\, r\, d\phi = I$$

$$2\pi r H_\phi = I$$

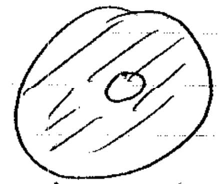
$$H_\phi = \frac{I}{2\pi r}$$

$$\overline{H} = \frac{I}{2\pi r} \hat{\phi}\, e^{-\gamma z}$$

add traveling wave parameters

Now apply L, C equations (Pozar example 2.1)

$$L = \frac{\mu}{|I|^2}\int_S \bar{H}\cdot\bar{H}^*\,dS$$



$$= \frac{\mu}{\cancel{|I|^2}}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\left(\frac{I}{2\pi r}e^{-\gamma z}\right)\hat{\phi}\cdot\hat{\phi}\left(\frac{I}{2\pi r}e^{-\gamma^* z}\right) r\,dr\,d\phi$$

$$= \mu\,\frac{2\pi}{(2\pi)^2}\ln(b/a)$$

$$L = \frac{\mu}{2\pi}\ln(b/a)\quad\left(\frac{H}{m}\right)$$

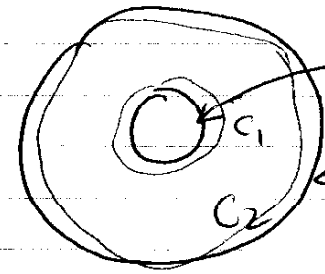
$$C = \frac{\varepsilon}{|V_0|^2}\int_S \bar{E}\cdot\bar{E}\,ds$$

$$= \frac{\varepsilon}{V^2}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\left(\frac{V}{r\ln b/a}\right)^2 r\,dr\,d\phi$$

$$= \frac{\varepsilon}{V^2}\,\frac{V^2}{(\ln b/a)^2}\,2\pi\ln b/a$$

$$C = 2\pi\varepsilon/\ln(b/a)\quad(F/m)$$

Resistance (R)



positive current on C_1

negative current on C_2

surface resistivity Ω/m^2

$$R = \frac{R_s}{|I|^2} \int_{C_1+C_2} \bar{H} \cdot \bar{H}^* \, d\ell$$

$$= \frac{R_s}{(2\pi)^2} \left[\int_{\phi=0}^{2\pi} \frac{1}{a^2} \overset{r}{a} \, d\phi + \int_{\phi=0}^{2\pi} \frac{1}{b^2} \, b \, d\phi \right]$$

$$= \frac{\overset{\frac{1}{\sigma \delta_s}}{R_s}}{2\pi} \left(\frac{1}{a} + \frac{1}{b} \right) \quad \frac{\Omega}{m} \qquad \Bigg\| \quad R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

Ulaby Table 2.

Conductance (G)

Time av power dissipated in lossy dielectric

$$P_d = \frac{\omega \varepsilon''}{2} \int_s \bar{E} \cdot \bar{E}^* \, ds = G \frac{V^2}{2}$$

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0 \varepsilon_r - j \frac{\sigma}{\omega}$$

$$G = \frac{\omega \varepsilon''}{|V|^2} \int_s \bar{E} \cdot \bar{E}^* \, ds$$

$$G = \frac{\omega \varepsilon''}{[\ln(b/a)]^2} \int_{\phi=0}^{2\pi} \int_{r=a}^{b} \frac{1}{r^2} \, r\,dr\,d\phi$$

$$= \frac{2\pi\omega\varepsilon''}{\ln b/a} \quad \frac{S}{m} \leftarrow 1/\Omega$$

$$\varepsilon'' = \frac{\sigma}{\omega}$$