The complex Hurwitz test for the analysis of spontaneous self-excitation in induction generators

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***Abstract*—Spontaneous self-excitation in induction generators is a fascinating phenomenon triggered by the instability of a zero equilibrium state. Prediction of this condition for various values of free parameters requires many computations of the eigenvalues of a 6×6 matrix over a large space. The paper uses a novel approach to stability using a transformation of the state-space system and an extension of the Hurwitz test to polynomials with complex coefficients. The analytic formulas that are obtained give the values of the minimum load resistance, the range of capacitor values, and the range of speeds for which spontaneous self-excitation appears. The paper concludes with an illustration of the results on an example.**

Keywords: induction machines, electric generators, self-excitation, unstable systems, complex Hurwitz test.

I. Introduction

The paper discusses the Hurwitz test for polynomials with complex coefficients and its application to the stability analysis of a class of linear systems. The complex Hurwitz test is an old result of the literature [8], possibly not well-known due to the lack of recognized applications. Indeed, the characteristic polynomials of real systems have real coefficients. However, the paper shows that a limited class of linear systems can be analyzed as complex systems having half the dimension of the original system. In such cases, the order of the characteristic polynomial can be cut in half and the complexity of the Hurwitz test can be significantly reduced, despite the increase in difficulty associated with the complex coefficients. In particular, the method is applicable to general symmetrical electric machines, including squirrel-cage and wound-rotor induction machines.

To illustrate the result, the paper considers the specific application to self-excited induction generators. Induction generators have gained interest in recent years because of their ruggedness and low-cost, and applicability in renewable energy applications [10], [13]. The understanding of self-excitation is important for squirrel-cage induction generators operated off-grid, but also for those connected to the grid, due to the need to protect the machines from overspeeding and overvoltages when accidentally disconnected [7].

Self-excitation of induction generators is an unusual phenomenon in electric machines. There always exists a zero equilibrium state that generates no power. Often, this state is stable, so that transfer to a non-zero steady-state requires triggering through pre-charged capacitors. In some fortunate cases, instability of the zero state triggers a departure towards a non-zero state of operation. We refer to such situation as *spontaneous self-excitation.* It is an unusual problem, where *instability is wanted* to achieve the desired result. The transient phenomenon of spontaneous self-excitation is nicely described in [9], where it is shown that the condition is related to the existence of unstable eigenvalues in a 6×6 matrix. Due to the size of the matrix, analytic conditions for stability are not found in [9], and the determination of stability has been performed numerically. While the computations can be performed rapidly with modern computers, the search may span a large space, with various parameters such as load resistance, capacitor values, and speed to be varied, and no *a priori* knowledge of the location, shape and number of the possible unstable regions. The contribution of this paper is to show that, by using the novel approach, analytic conditions for stability can be obtained. As a result, self-contained formulas give an understanding of exactly when spontaneous self-excitation occurs, and how it is affected by various parameters. The paper builds on the original results of [4], adding a more general presentation of systems for which the results can be applied, fixing some small errors, and providing further explanations.

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II. The complex Hurwitz test and its application for stability analysis

The classical Hurwitz test is a fundamental result that is well known to control engineers in the form of the Routh-Hurwitz criterion. The extension of the test to polynomials with complex coefficients is not as well known, and is given below [8].

Lemma 1 - Complex Hurwitz test: The roots of the polynomial

$$P(s) = s^n + c_1 s^{n-1} + c_2 s^{n-2} + ... + c_n \tag{1}$$

with complex coefficients

$$c_k = a_k + jb_k, \quad k = 1, \ldots, n \tag{2}$$

are in the open left-half plane if and only if $\Delta_k > 0$ for

$k = 1, ..., n$, where

$$\Delta_k = \begin{vmatrix} a_1, & a_3, & a_5, & \dots, & a_{2k-1}, & -b_2, & -b_4, & \dots, & -b_{2k-2} \\ 1, & a_2, & a_4, & \dots, & a_{2k-2}, & -b_1, & -b_3, & \dots, & -b_{2k-3} \\ & & & \dots, & & & & \dots, & \\ 0, & & & \dots, & a_k, & 0, & & \dots, & -b_{k-1} \\ 0, & b_2, & b_4, & \dots, & b_{2k-2}, & a_1, & a_3, & \dots, & a_{2k-3} \\ 0, & b_1, & b_3, & \dots, & b_{2k-3}, & 1, & a_2, & \dots, & a_{2k-4} \\ & & & \dots, & & & & \dots, & \\ 0, & & & \dots, & b_k, & 0, & & \dots, & a_{k-1} \end{vmatrix} \quad (3)$$

and $a_r = b_r = 0$ for $r > n$.$\square$

Although systems with complex coefficients are not typically encountered in control engineering, some real systems can be transformed into smaller systems with complex coefficients where the application of the lemma has certain advantages. Indeed, consider the state-space system

$$E\dot{x} = Fx \quad (4)$$

whose stability is determined by the location of the roots of

$$\det(A_R(s)) = \det(Es - F) \quad (5)$$

Assume that $A_R(s)$ has the specific structure

$$A_R(s) = \begin{pmatrix} A_{11}(s) & -A_{21}(s) \\ A_{21}(s) & A_{11}(s) \end{pmatrix} \quad (6)$$

and define the matrix

$$T_C = \begin{pmatrix} I & jI \end{pmatrix} \quad (7)$$

where I is the identity matrix with dimension n and $2n$ is the dimension of the state-space system. One has that

$$T_C A_R(s) = \begin{pmatrix} A_C(s) & jA_C(s) \end{pmatrix} \quad (8)$$

where

$$A_C(s) = A_{11}(s) + jA_{21}(s) \quad (9)$$

Note that the polynomial $\det(A_R(s))$ has $2n$ roots and, because its coefficients are real, the roots must either be real, or appear as complex pairs. On the other hand, the polynomial with complex coefficients $\det(A_C(s))$ has degree n, but its roots can lie anywhere in the complex plane. The following fact shows that the roots of the two polynomials are closely connected.

Fact 1: Any root of $\det(A_C(s)) = 0$ is a root of $\det(A_R(s)) = 0$. On the other hand, if s_0 is a root of $\det(A_R(s)) = 0$, then either s_0 or its complex conjugate s_0^* is a root of $\det(A_C(s)) = 0$.

Proof of Fact 1:

Part 1: If $\det(A_C(s_0)) = 0$, there exists $z_A \in C^n$, such that $z_A \neq 0$ and $z_A^T A_C(s_0) = 0$. Letting $z_0^T = z_A^T T_C = \begin{pmatrix} z_A^T & jz_A^T \end{pmatrix}$, (8) shows that $z_0 \neq 0$ and $z_0^T A_R(s_0) = 0$. Therefore, $\det(A_R(s_0)) = 0$.

Part 2: If $\det(A_R(s_0)) = 0$, there exists $z_0 \in C^{2n}$, such that $z_0 \neq 0$ and $A_R(s_0)z_0 = 0$. Let $z_A \in C^n$, $z_B \in C^n$ such that $z_0^T = (z_A^T, z_B^T)$. Then, (8) implies that $A_C(s_0)(z_A + jz_B) = 0$. If $z_A + jz_B \neq 0$, it follows that $\det(A_C(s_0)) = 0$. On the other hand, if $z_A + jz_B = 0$, one must have $z_A - jz_B \neq 0$, or else $z_0 = 0$. If $z_A - jz_B \neq 0$, $A_C(s_0^*)(z_A - jz_B) = 0$, which implies that $\det(A_C(s_0^*)) = 0$. $\square$

The proof of Fact 1 can also be derived from the property that the eigenvalues of a matrix $[A\ -B;\ B\ A]$ are the same as those of the matrix $[A + jB\ 0;\ 0\ A - jB]$, which was used in [1]. A consequence of the special structure of (6) is that the roots of $\det(A_R(s)) = 0$ must be either complex pairs or double real pairs. In other words, there cannot be single real roots. Further, each root of $\det(A_C(s)) = 0$ is one of the roots in a pair of roots of $\det(A_R(s)) = 0$. Thus, $A_C(s)$ contains the full information about the dynamics of the original system: all the poles of the original system can be obtained for the roots of $\det(A_C(s))$. Also, the roots of $\det(A_R(s))$ are in the open left-half plane if and only if the roots of $\det(A_C(s))$ are in the open left-half plane.

Given these properties, the stability of the original state-space system can be determined by using either the real Hurwitz test on $\det(A_R(s))$ or the complex Hurwitz test on $\det(A_C(s))$. In general, the complex Hurwitz test is more complicated for an n^{th} order polynomial than the real Hurwitz test for an n^{th} order polynomial, but simpler than the real Hurwitz test for a $2n^{th}$ order polynomial. Therefore, the approach can have advantages for systems that satisfy the symmetry properties (6) and are of sufficiently low order to be tractable. Although such systems are rare, several examples have been found in the analysis of induction machines. We focus here on spontaneous self-excitation in induction generators.

III. Application to spontaneous self-excitation

A. Model and problem formulation

Consider the following model of a two-phase induction generator

$$\begin{aligned} L_S \frac{di_{SA}}{dt} + R_S i_{SA} + M \frac{di_{RA}}{dt} &= v_{SA} \\ L_S \frac{di_{SB}}{dt} + R_S i_{SB} + M \frac{di_{RB}}{dt} &= v_{SB} \\ M \frac{di_{SA}}{dt} + n_P \omega M i_{SB} + L_R \frac{di_{RA}}{dt} & \\ + R_R i_{RA} + n_P \omega L_R i_{RB} &= 0 \\ -n_P \omega M i_{SA} + M \frac{di_{SB}}{dt} - n_P \omega L_R i_{RA} & \\ + L_R \frac{di_{RB}}{dt} + R_R i_{RB} &= 0 \end{aligned} \quad (10)$$

where v_{SA}, v_{SB} are the stator voltages, i_{SA}, i_{SB} are the stator currents, i_{RA}, i_{RB} are the rotor currents transformed into the stator frame of reference (or equivalent rotor currents in the case of a squirrel-cage generator), and ω is the speed of the generator. For the purpose of the analysis of this paper, the speed is assumed constant. The parameters of the generator are L_S, the stator inductance, L_R, the rotor inductance, M, the mutual inductance between the stator and rotor windings, R_S, the stator resistance, R_R, the rotor resistance, and n_P, the number of pole pairs. In the case of a three-phase motor, a three-phase to two-phase transformation should be used first to apply the results.

Attached to each stator winding is a load, as well as a capacitor C that is added to provide the required reactive power. The load is assumed to be purely resistive, with resistance R_L. The capacitor C is placed in parallel with the load. For convenience, we derive the results in terms of the admittance $Y_L = 1/R_L$. This is simply so that the no-load case corresponds to $Y_L = 0$ instead of $R_L = \infty$. We have

$$\begin{aligned} C\frac{dv_{SA}}{dt} + i_{SA} + Y_L v_{SA} &= 0 \\ C\frac{dv_{SB}}{dt} + i_{SB} + Y_L v_{SB} &= 0 \end{aligned} \tag{11}$$

As was observed in [9], it is possible for one or more poles of the system to lie in the right-half plane. In this case, although a zero solution exists, it is unstable. Then, a non-zero initial state of arbitrary value will result in a growth of the voltages and currents, until magnetic saturation is encountered. A limit cycle of the nonlinear system results, and production of AC power is possible. The analysis of the complete self-excitation phenomenon requires a complicated nonlinear model with inductances depending on the current vector. However, the onset of instability discussed in this paper can be analyzed with the model of the system linearized around the zero state. This model is described by (10), with the inductances corresponding to zero currents.

The prediction of spontaneous self-excitation can be achieved by computing the roots of the determinant of the 6×6 matrix. This computation can be brought into an equivalent eigenvalue problem. The limitation of the approach, however, is that the result can only be obtained numerically. The computation may have to be performed for various capacitor values and speeds, as well as load resistance if it is a free parameter. Whether one or more regions are possible is unknown. In theory, stability conditions could be derived by applying the Routh-Hurwitz test to the characteristic polynomial $\det(A_R(s))$. Such task, however, is very complicated, given the dimension of the problem (6×6 matrix and 6^{th} order polynomial).

B. Self-excitation conditions based on the complex Hurwitz test

The application of the complex Hurwitz test requires consideration of the equivalent complex matrix $A_C(s)$ given by

$$A_C(s) = \begin{pmatrix} sL_S + R_S & sM & -1 \\ sM - jn_P\omega M & sL_R + R_R - jn_P\omega L_R & 0 \\ 1 & 0 & sC + Y_L \end{pmatrix} \tag{12}$$

Specifically, the matrix $A_R(s)$ has the structure (6) with $A_{11}(s) = \mathrm{Re}(A_C(s))$ and $A_{21}(s) = \mathrm{Im}(A_C(s))$. The poles are the roots of the third-order polynomial with complex coefficients

$$\begin{aligned} \det(A_C(s)) = a_0 s^3 + (a_1 - jn_P\omega d_1)s^2 \\ +(a_2 - jn_p\omega d_2)s + (a_3 - jn_P\omega d_3) \end{aligned} \tag{13}$$

where

$$\begin{aligned} a_0 &= C(L_S L_R - M^2) \\ a_1 &= Y_L(L_S L_R - M^2) + C(L_S R_R + L_R R_S) \\ a_2 &= Y_L(L_S R_R + L_R R_S) + (C R_S R_R + L_R) \\ a_3 &= R_R(Y_L R_S + 1) \\ d_1 &= a_0 \\ d_2 &= Y_L(L_S L_R - M^2) + C L_R R_S \\ d_3 &= L_R(Y_L R_S + 1) \end{aligned} \tag{14}$$

The machine parameters are all positive and the *leakage factor*

$$\sigma = \frac{L_S L_R - M^2}{L_S L_R} \tag{15}$$

is such that $1 > \sigma > 0$. It follows that $a_i > 0$ and $d_j > 0$ for all applicable i and j. Although $a_0 \neq 1$, since $a_0 > 0$, lemma 1 can be applied by replacing the entries equal to 1 in (3) by a_0. Further, the first test variable of the complex Hurwitz test is $\Delta_1 = a_1 > 0$. Therefore, it can be shown that the complex Hurwitz test reduces to the following two conditions

$$\begin{aligned} \Delta_2 &= a_1(a_1 a_2 - a_3 d_1) \\ &\quad + d_1 d_2 (a_1 - d_2)(n_P\omega)^2 \\ \Delta_3 &= \alpha(n_P\omega)^4 + \beta(n_P\omega)^2 + \gamma \end{aligned} \tag{16}$$

where

$$\begin{aligned} \alpha &= (a_1 - d_2) d_1 d_2^2 d_3 \\ \beta &= a_1^2 a_2 d_2 d_3 - a_1^3 d_3^2 - 3a_1 a_3 d_1 d_2 d_3 \\ &\quad + 2a_1^2 a_3 d_1 d_3 + a_1 a_2 a_3 d_1 d_2 - a_1 a_3^2 d_1^2 \\ &\quad - a_2 a_3 d_1 d_2^2 + a_3^2 d_1^2 d_2 \\ \gamma &= a_3(a_1 a_2 - a_3 d_1)^2 \end{aligned} \tag{17}$$

With these preliminaries, the following fact can be derived.

Fact 2: Spontaneous self-excitation occurs if and only if the parameters of the induction generator satisfy

$$\beta < -2\sqrt{\alpha\gamma} \tag{18}$$

In this case, spontaneous self-excitation occurs for a single range of speeds $\omega \in (\omega_{\min}, \omega_{\max})$, such that

$$\begin{aligned} \omega_{\min} &= \frac{1}{n_P}\sqrt{\frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}} \\ \omega_{\max} &= \frac{1}{n_P}\sqrt{\frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}} \end{aligned} \tag{19}$$

and the system can only have one pair of unstable poles in the self-excitation region.

Proof of Fact 2: Since

$$a_1 - d_2 = C L_S R_R \tag{20}$$

the second term of Δ_2 is positive. As for the first term of Δ_2, one obtains, after simplifications

$$\begin{aligned}a_1a_2 - a_3d_1 = \\ &Y_L^2(L_SL_R - M^2)(L_SR_R + L_RR_S) \\ &+ Y_L(L_SL_R - M^2)L_R \\ &+ CY_L(L_SR_R + L_RR_S)^2 \\ &+ C^2R_SR_R(L_SR_R + L_RR_S) \\ &+ CL_R^2R_S + CR_RM^2 \end{aligned} \tag{21}$$

which is positive. One may conclude that Δ_2 is always positive and that the stability of the generator is determined solely by the condition $\Delta_3 > 0$. Given that $a_1 - d_2 > 0$, $\alpha > 0$ and $\gamma > 0$, a necessary condition for instability is that the parameter β must be negative. If it is the case, one also needs to find a speed such that $\Delta_3 < 0$. Note that Δ_3 is a quadratic function of $(n_P\omega)^2$, which is positive for $\omega^2 = 0$ and for large ω^2. Therefore, there must be two real positive roots of $\Delta_3\left((n_P\omega)^2\right) = 0$, in order to have a range of speeds for which $\Delta_3 < 0$. The condition on β (18) is necessary and sufficient for this to be the case.

The speed range is obtained from the solutions of the quadratic equality $\Delta_3 = 0$. The number of unstable roots can be predicted because the Hurwitz array, whose leading column is composed of Δ_1, Δ_2, Δ_3, specifies the number of right-half plane roots as for the classical Hurwitz test. Since there can only be one sign change, the system with complex poles can only have one unstable pole. Therefore, the original system can only have one pair of unstable complex poles or two identical real poles. In practice, the case with two real poles is not encountered. □

The result of Fact 2 is enlightening, because it shows that the origin of spontaneous self-excitation is a growing oscillation whose rate of growth is determined by the real part of a complex pole, and whose frequency is determined by the imaginary part of the pole. There cannot be two competing oscillations of different frequencies. The overall power of the result is that it gives a direct computation of the speed range for which spontaneous self-excitation will occur. The critical speeds are obtained by solving a single quartic equation, rather than computing the eigenvalues of a 6×6 matrix for a large number of speeds. The effect of various parameters can more easily and rapidly be assessed. However, it should be noted that, as with numerical approach, the analysis assumes that the speed is constant.

C. Self-excitation conditions based on singularity

Since instability is caused by the crossing of a single root of the complex polynomial $\det(A_C(s))$ across the imaginary axis, a condition for spontaneous self-excitation is that, for some ω_e,

$$\det\left(A_C(j\omega_e)\right) = 0 \tag{22}$$

The real and imaginary parts of (22) yield the following two conditions

$$\begin{gathered} Y_LR_SR_R - \omega_e^2CR_RL_S + R_R = \\ (\omega_e - n_P\omega)\omega_e\left(Y_L(L_SL_R - M^2) + R_SL_RC\right), \\ \omega_e\left(R_RL_SY_L + CR_SR_R\right) = (\omega_e - n_P\omega) \\ \left(\omega_e^2C(L_SL_R - M^2) - R_SL_RY_L - L_R\right) \end{gathered} \tag{23}$$

Therefore, stability boundaries can be obtained from the solutions of these equations, as stated in the following fact.

Fact 3: Spontaneous self-excitation is possible if and only if

$$R_L > \frac{4\sqrt{\sigma}}{\left(\sqrt{\sigma} - 1\right)^2}R_S \tag{24}$$

If R_L satisfies (24), the range of capacitor values for which self-excitation occurs is given by the limits

$$\begin{aligned} C_{\min} &= \frac{-g_2 - \sqrt{g_2^2 - 4g_1g_3}}{2g_1}, \\ C_{\max} &= \frac{-g_2 + \sqrt{g_2^2 - 4g_1g_3}}{2g_1} \end{aligned} \tag{25}$$

where

$$\begin{aligned} g_1 &= R_S^2 \\ g_2 &= 2L_S(R_SY_L + 1)\sqrt{\sigma} - (\sigma + 1)L_S \\ g_3 &= \sigma L_S^2Y_L^2 \end{aligned} \tag{26}$$

For any value of C in the range defined by (25), the range of electrical frequencies for which self-excitation occurs is given by the limits

$$\begin{aligned} \omega_{e,\min} &= \sqrt{\frac{-f_2 - \sqrt{f_2^2 - 4f_1f_3}}{2f_1}} \\ \omega_{e,\max} &= \sqrt{\frac{-f_2 + \sqrt{f_2^2 - 4f_1f_3}}{2f_1}} \end{aligned} \tag{27}$$

where

$$\begin{aligned} f_1 &= C^2L_S(L_SL_R - M^2) \\ f_2 &= Y_L^2L_S(L_SL_R - M^2) + C^2R_S^2L_R \\ &\quad - C(2L_SL_R - M^2) \\ f_3 &= L_R(Y_LR_S + 1)^2 \end{aligned} \tag{28}$$

The range of mechanical speeds for which self-excitation occurs is obtained by replacing ω_e by $\omega_{e,\min}$ and $\omega_{e,\max}$ in the following equation

$$\omega = h_1\omega_e - h_2/\omega_e \tag{29}$$

where

$$\begin{aligned} h_1 &= \frac{Y_L(L_SL_R - M^2) + C(R_SL_R + R_RL_S)}{n_P\left(Y_L(L_SL_R - M^2) + CR_SL_R\right)} \\ h_2 &= \frac{R_R(1 + Y_LR_S)}{n_P\left(Y_L(L_SL_R - M^2) + CR_SL_R\right)} \end{aligned} \tag{30}$$

Proof of Fact 3: The first equation of (23) leads to (29). Eliminating $\omega_e - n_P\omega$ from the two equations gives, after simplifications, the quartic equation in ω_e

$$f_1\omega_e^4 + f_2\omega_e^2 + f_3 = 0 \tag{31}$$

where f_1, f_2, and f_3 are given by (28), and f_1 and f_3 are both positive. The quartic equation is a quadratic equation in ω_e^2, which has a positive real root if and only if it has two positive real roots (given that $f_1 > 0$ and $f_3 > 0$). Thus, a solution exists for ω_e if and only if

$$f_2 < -2\sqrt{f_1 f_3} \tag{32}$$

If the condition is satisfied, the solutions for ω_e are given by (27). The speed ω is related to the electrical frequency ω_e by either equation of (23). The first equation gives (29), with $h_1 > 0$, $h_2 > 0$. Therefore, ω increases monotonically with ω_e, and limits for ω can be obtained from limits for ω_e using (29). The fact that the interval between the limits corresponds to self-excitation is known from Fact 2.

After simplifications, (32) gives the following inequality

$$g_1 C^2 + g_2 C + g_3 < 0 \tag{33}$$

where the parameters g_1, g_2, and g_3 are given by (26). Given that $g_1 > 0$ and $g_3 > 0$, we again find ouselves in a situation where the inequality can have a solution if and only if the quadratic equality has two positive real roots, which requires that

$$g_2 < -2\sqrt{g_1 g_3} \tag{34}$$

If the condition is sastisfied, the roots of the quadratic equality associated with (33) are given by (25) and specify the range of capacitor values. After simplifications, (34) gives

$$\frac{\sigma + 1 - 2\sqrt{\sigma}}{4\sqrt{\sigma}} > R_S Y_L \tag{35}$$

which yields (24). It is interesting to note that the minimum load resistance is only dependent on two parameters: the stator resistance and the leakage factor. □

D. Comparison between the stability and singularity analyses

The approach using the singularity condition (22) is similar to the derivation of steady-state conditions for self-excited induction generators except that, in the latter case, the differential equations are those obtained from linearization around some equilibrium state in the magnetic saturation region. The singularity approach is simpler and gives useful results that are not directly available from the stability approach. However, the regions of stability cannot be unambiguously determined from the boundaries for stability. Even knowing that the system is stable at low speeds, it could be possible for one pole to cross the imaginary axis at the lower speed while, at the higher speed, either a second pole would become unstable, or the first would become stable again. The fact that instability occurs *between* the speeds is only known from the stability analysis.

It is also interesting to note that the equations do not naturally develop in identical forms, and provide answers to different questions. In the stability approach, the speed range is obtained as (19). In the singularity approach, the electrical frequencies are first obtained, and then the speed range is determined using (29). However, the results are equivalent, a fact that can be verified as follows. The two equations of (23) are identical to

$$\begin{aligned} a_1 \omega_e^2 - n_P \omega d_2 \omega_e - a_3 &= 0 \\ d_1 \omega_e^3 - n_P \omega d_1 \omega_e^2 - a_2 \omega_e + n_P \omega d_3 &= 0 \end{aligned} \tag{36}$$

The two polynomials have a common root if and only if their resultant (*i.e.,* the determininant of the associated Sylvester matrix of dimension 5) is equal to zero. Symbolic computations performed by the authors and not reproduced here show that the resultant gives the same polynomial as Δ_3 in (16). Based on this result, one can also expect that (32) must be equivalent to (18), although the fact is not immediately obvious.

Note that both approaches only address the onset of self-excitation from the zero state. To understand the steady-state regimes, a more complicated model accounting for nonlinear magnetic saturation must be used [5]. Unfortunately, the stability of the power-producing operating regime cannot be assessed using the stability approach of this paper because the state-space model does not satisfy the required symmetry conditions for a state vector other than zero. Interestingly, the singularity approach can be used to determine non-zero steady-states by replacing the fixed inductances with nonlinear functions of the currents.

E. Example

Consider the generator of [12], with $R_S = 1.7\Omega$, $R_R = 2.7\Omega$, $L_S = L_R = 191.4mH$, and $M = 180mH$. Let $R_L = \infty$ and $C = 300\mu F$. The stability conditions give $\omega_{\min} = 66.7$ rad/s and $\omega_{\max} = 465.3$ rad/s. The minimum speed of 640 rpm is consistent with Fig. 2 of the paper [12]. At $\omega = 100$ rad/s, the roots of $\det(A_C(s))$ are given by $-73.4 - 377.9j$, $-142.1 + 389.1j$, and $16.6 + 188.7j$. The roots of $\det(A_R(s))$ are given by $-73.4 \pm 377.9j$, $-142.1 \pm 389.1j$, and $16.6 \pm 188.7j$, with the last pair being the unstable pair of poles leading to spontaneous self-excitation. For $R_L = 25\Omega$, the range is reduced to $86.9 - 265.2$ rad/s and the range vanishes for $R_L < 14.5\Omega$. Lower resistance values require a larger capacitor although, for low enough resistance, no capacitor produces spontaneous self-excitation.

Using the results, one may plot the combinations of speed and capacitor values were spontaneous self-excitation occurs. Fig. 1 shows the upper and lower limits of speed as functions of the capacitor value, for several values of load resistance. The figure is also consistent with the figures of [12].

The singularity conditions give a minimum value of $R_L = 5.3064\Omega$. For this extreme case, (25) gives both minimum and maximum values of the capacitor as $C = 7.2131mF$. For this value of capacitor, the range of electrical frequencies is also a single frequency $\omega_e = 53.04$ rad/s, and the mechanical frequency is $\omega = 47.27$ rad/s. This is a limit case. Note that the slip $\omega_e - n_P\omega$ (with $n_P = 2$) is -41.5 rad/s, which is negative, as must be for the generator mode. For $R_L = \infty$, (25) gives $C_{\min} = 0$ and $C_{\max} = 28.9$ mF. For $C_{\max}$, (27) gives a single frequency $\omega_e = 23.1$ rad/s, and a mechanical frequency $\omega = 23.64$ rad/s. Again, this is a limit case. The

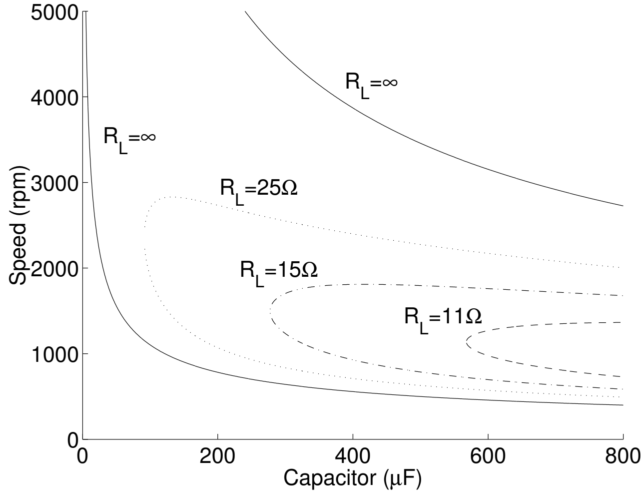

Fig. 1. Critical speeds for self-excitation

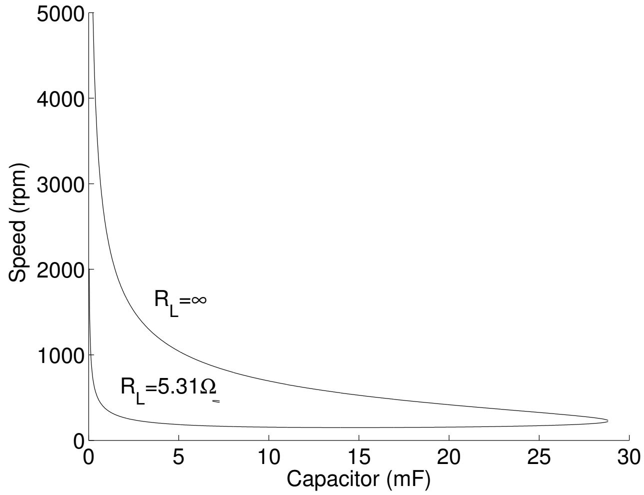

Fig. 2. Critical speeds for self-excitation (expanded view)

results are shown on Fig. 2, which is an expanded view of Fig. 1. The region of spontaneous self-excitation for infinite load resistance is delimited by the outside curve, while the region for $R_L = 5.31\Omega$ (which is slightly greater than the minimum resistance) is inside the tiny spot under the Ω symbol on Fig. 2.

IV. Acknowledgment

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V. Conclusions

The paper presented the complex Hurwitz test and its application to a special class of linear systems. Using the approach, analytic conditions were found for spontaneous self-excitation in induction generators. The formula conveniently replace the exhaustive numerical search that was required before, and provided the justification for a second approach based on the singularity test of a complex matrix. The singularity approach provided other interesting results that were illustrated on an example. Although the range of applications of the stability approach is limited, other examples have been found in the analysis of induction machines. Specifically, a closed-loop control algorithm for doubly-fed induction generators was considered, for which a proof was not previously found possible [2]. The complex Hurwitz test approach provided a simple formula that specified the relationship to be satisfied by the PI gains for stability [6]. The application of the Hurwitz test was also considered for another control law for doubly-fed induction generators [11]. In this case, the derivations were found tedious and better performed using a symbolic computational engine [3]. Doing so with the *Symbolic Toolbox* of *Matlab,* expressions for Δ_2 and Δ_3 were obtained having all positive terms, confirming the results of [11] that used a Lyapunov function. The Hurwitz test was not preferable to the Lyapunov method: it simply eliminated the guesswork in finding such a Lyapunov function. Note that the complex representation of symmetric induction machines has often been used in the literature, but without being used for stability analysis. A contribution of this paper is to show that the Hurwitz test, well-known but only in the real domain, can in fact be used for such purpose.

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